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We study the localization of collective pair excitations in weakly-interacting Bose superfluids in one-dimensional quasiperiodic lattices. The localization diagram is first determined numerically. For intermediate interaction and quasiperiodic amplitude we find a sharp localization transition, with extended low-energy states and localized high-energy states. We then develop an analytical treatment, which allows us to quantitatively map the localization transition into that of an effective multi-harmonic quasiperiodic system.

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Quasiperdiodic systems, which are formed of a small number of incommensurate sinusoidal components, constitute an appealing intermediate between disordered and periodic systems. Such structures appear naturally in the growth of certain crystals [1] or as a result of charge-density waves [2]. They can also be created on purpose in crystals by molecular epitaxy [3] or with ultracold atoms in optical potentials [4, 5]. In quasiperiodic systems, the lack of translation invariance can induce localization of linear waves, similarly as the phenomenon of Anderson localization in disordered systems [6]. In quasiperiodic systems, however, the quasi-repetition of finite patterns radically changes the localization picture. For instance, in a one-dimensional disordered system, any quantum particle is localized with an energy-dependent localization length [7, 8]. In contrast, for a quasiperiodic system made of a single incommensurate sinusoidal modulation of a main periodic lattice, there is a localization transition for some critical strength of the quasiperiodic component beyond which the states are localized with a localization length that is independent of the energy [9].

The extension of the concept of localization to interacting quantum systems is attracting a considerable attention as regards phase diagrams [10], many-body localization transitions [11], and localization of collective excitations [12–14]. These issues have been first investigated for purely disordered systems and extensions to quasiperiodic systems are just starting. So far, most studies focused on the phase diagram of one-dimensional bosons in quasiperiodic lattices at zero temperature [15, 16], finite temperature [17], and infinite temperature [18]. Conversely, the localization of collective excitations remains largely open. This issue is particularly important because the transport of collective excitations governs many dynamical effects in correlated quantum systems [19], for instance the propagation of correlations in recently-developed quench experiments [20].

Here we study the localization of collective pair excitations in weakly-interacting Bose superfluids subjected to a one-dimensional quasiperiodic lattice. We first determine the localization diagram numerically and show that, for intermediate interaction and quasiperiodic amplitude, there is a sharp localization transition, separating bands of states that are extended at low energy and localized at high energy. We then develop an analytical treatment, which allows us to reproduce the numerical results accurately. Moreover, it allows us to quantitatively map the localization transition into that of an effective multi-harmonic quasiperiodic system.

The starting point of our study is the Aubry-André-Hubbard Hamiltonian,

$$\hat{H} = -\sum_{j,l} T_{j,l}\hat{a}_j^\dagger \hat{a}_l + \sum_j V_j \hat{a}_j^\dagger \hat{a}_j + \frac{U}{2}\sum_j \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j, \quad (1)$$

which governs the low-energy physics of interacting bosons in one-dimensional (1D) quasiperiodic lattices. In Eq. (1), $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the bosonic annihilation and creation operators at the lattice site j. The first term represents quantum tunneling with the hopping matrix $\hat{T}$, which includes nearest-neighbor tunnelling, $T_{j,j\pm1} = t$ and $T_{j,l} = 0$ for $|j-l| > 1$, as well as the homogeneous on-site term $T_{j,j} = -2t$, for convenience. Within this convention, the free-particle spectrum, $\varepsilon_k = 4t\sin^2(k/2)$, is centered on $\varepsilon = 2t$ with the band edges $\varepsilon = 0$ and $\varepsilon = 4t$. The second term represents the on-site quasiperiodic potential modulation, $V_j = \Delta\cos(2\pi r j + \varphi)$, where φ is a phase, Δ is the quasiperiodic amplitude, and r is an irrational number. The third term represents on-site repulsive interactions with the interaction energy $U > 0$.

In the weakly-interacting regime with high occupation number per lattice site ($n \gg U/t$, with n the mean density), we can rely on meanfield theory [21]. A similar approach has been presented elsewhere for disordered systems in continuous [14, 22, 23] or lattice [24] spaces, and we just outline it here. The density background n_j is first determined by minimizing the classical energy functional, obtained by replacing the operator $\hat{a}_j$ by the real-valued field $\phi_j \equiv \sqrt{n_j}$ in Eq. (1). It yields the Gross-Pitaevskii

equation (GPE),

$$\mu\phi_j = -\hat{T}\phi_j + V_j\phi_j + U\phi_j^3 \qquad (2)$$

where the term $\hat{T}\phi_j$ is a shortcut for the hopping matrix contribution $t(\phi_{j+1} - 2\phi_j + \phi_{j-1})$ and μ is the chemical potential. The collective pair excitations of the Bose superfluid, which are represented by two fields u_j and v_j, are then found by expanding Hamiltonian (1) up to second order in the Bogoliubov operator $\delta\hat{n}_j/2\sqrt{n_j} + i\sqrt{n_j}\delta\hat{\theta}_j$, where $\delta\hat{n}_j$ and $\delta\hat{\theta}_j$ are the density and phase fluctuation operators, and diagonalizing the resulting quadratic Hamiltonian. The excitation energy E and wavefunctions u_j and v_j are the solutions of the Bogoliubov de-Gennes equations (BdGEs)

$$\begin{bmatrix} -\hat{T}+V_j-\mu+2Un_j & Un_j \\ -Un_j & \hat{T}-V_j+\mu-2Un_j \end{bmatrix} \begin{bmatrix} u_j \\ v_j \end{bmatrix} = E \begin{bmatrix} u_j \\ v_j \end{bmatrix}. \quad (3)$$

Equations (2) and (3) form the complete set to determine the elementary excitations of the Bose fluid in the quasiperiodic lattice.

We first solve Eqs. (2) and (3) numerically in the 1D quasiperiodic lattice. The number of lattice sites is chosen to be a Fibonacci number F_p and r is taken as the ratio F_{p-1}/F_p. It allows us to use periodic boundary conditions and a good approximation of an incommensurate ratio $(\sqrt{5}-1)/2$ [9]. In practice, we use $F_p = 610$, which yields $r = (\sqrt{5}-1)/2 \pm 0.000002$. The density background is computed by solving the GPE using imaginary-time propagation with a Crank-Nicolson scheme [25]. The good numerical convergence of the imaginary-time propagation of the GPE is a delicate point for the subsequent determination of the collective excitations using Eq. (3). The convergence criterion applies to the effective, imaginary time-dependent chemical potential $\mu(\tau) \equiv -\frac{\hbar}{2}\frac{d}{d\tau}\log\left(\sum_j n_j\right)$. We have checked that the density profile is unaffected when the precision threshold varies from 10^{-8} to 10^{-15}. The same holds when the imaginary time step $\Delta\tau$ used in the propagation varies from $0.01/t$ to $0.5/t$. Moreover, the density profile precisely agrees with perturbative expansion of the GPE solution implemented up to order 50 [26]. All together, the precision on the density profile n_j is of the order of 10^{-8} for all results presented here. The excitations are then computed by exact diagonalization of the matrix in Eq. (3) using Lanczos algorithm for sparse non-Hermitian eigenproblems [25].

The numerical results are summarized on the diagram of Fig. 1(a). It displays three different regimes. For weak quasiperiodic amplitude Δ and strong interaction U, the density background is fully connected and all excitations are extended ('extended regime'). For a given interaction strength U and tunneling t, the density modulations increase with the quasiperiodic amplitude Δ. Above a critical value of Δ_c, the density profile gets fragmented

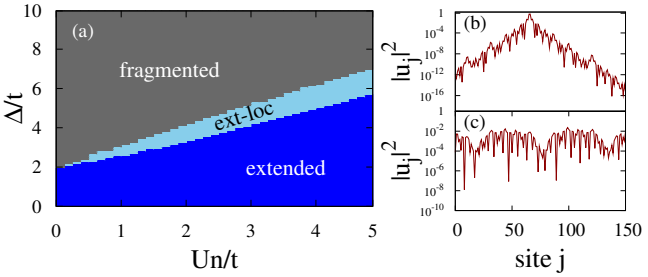

Figure 1. Numerical results. (a) Localization diagram as a function of the interaction strength and the quasiperiodic amplitude. It displays three regimes: (i) 'Extended regime' where the density background is connected and all excitations are extended; (ii) 'Fragmented regime' where the density background is fragmented; (iii) 'Extended-localized regime' where the density background is connected and the excitation spectrum shows a delocalization-localization transition with exponentially localized high-energy states and extended low-energy states. (b)-(c) Typical excitation wavefunction u in the localized (b) and extended (c) regimes, plotted in semi-log scale and for the 150 first lattice sites (similar plots are found for the v wavefunctions). The two panels correspond to two excitations with consecutive energies above (b) and below (c) the mobility edge for $Un/t = 1.75$ and $\Delta/t = 3.3$.

('fragmented regime'), which yields the upper boundary on the diagram. The fragmentation condition is chosen to be the minimal value of Δ such that at least one lattice site has a density lower than 0.01 atom per site. We have checked that varying this arbitrary threshold down to 0.001 yields insignificant changes of the fragmentation boundary. Moreover the latter is in good agreement with the experimental observation of Ref. [4]. In the fragmented regime, the density profile is cut in disconnected pieces. It corresponds to trivial localization, a case that we disregard in the following. Notice that in the limit $U \to 0$, we recover the critical value $\Delta_c = 2t$, that is the localization transition of the noninteracting Aubry-André model. The most interesting regime appears for intermediate quasiperiodic amplitude ('ext-loc regime'). In this regime, although the density background is fully connected, we find a localization transition of the collective excitations. Remarkably enough, they are the high-energy excitations that are exponentially localized over a few lattice sites [see Fig. 1(b)] while the low-energy excitations are extended over the whole system [see Fig. 1(c)]. This transition is sharp as exemplified on Figs. 1(b) and (c), which correspond to two excitations of consecutive energies for $Un = 1.75t$ and $\Delta = 3.3t$.

In order to characterize the localization transition, we compute two Lyapunov exponents for the excitations, which correspond to the two Bogoliubov wavefunctions, $\gamma_u(E) \equiv -\lim_{j\to\infty}\log|u_j|/j$ and $\gamma_v(E) \equiv -\lim_{j\to\infty}\log|v_j|/j$. They are extracted from fits in the tails of the logarithm of the wavefunctions u and v. Figure 2(a) displays those Lyapunov exponents versus the

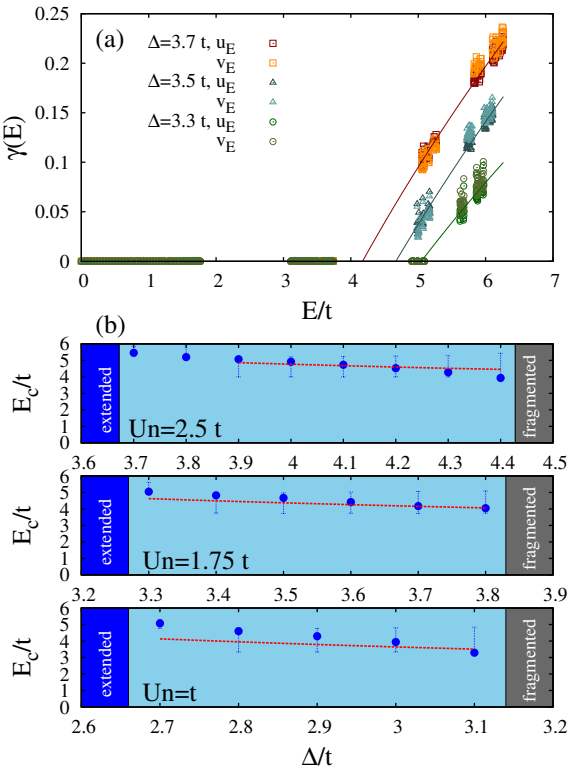

Figure 2. (a) Lyapunov exponents of the Bogoliubov wavefunctions u and v, for $Un/t = 1.75$ and $\Delta/t = 3.3, 3.5, 3.7$. The excitation spectrum is banded and displays a sharp localization transition separating extended ($\gamma = 0$) and localized ($\gamma > 0$) states. (b) Mobility edge as a function of the quasiperiodic amplitude Δ/t as extracted from power-law fits to the numerical $\gamma(E)$ curves [solid lines on panel (a)]. Error-like bars correspond to the edges of the minigap containing the mobility edge. The dotted, red line shows the analytical prediction of the locator theory applied to the effective model (8) with the potential (11).

excitation energy E, for fixed interaction and disorder strengths. The Lyapunov exponents γ_u and γ_v are indistinguishable and herafter we omit the wavefunction index u or v. In the 'ext-loc' regime the Lyapunov exponent curves clearly show the transition, separating extended ($\gamma = 0$) and localized ($\gamma > 0$) states. The excitation spectrum splits in several bands separated by minigaps, a general feature in quasiperiodic systems [9, 27–30]. The transition generally lies into one of the minigaps. To determine the mobility edge E_c, we thus rely on fits of the $\gamma(E)$ curves wih several fitting functionals [linear, $\gamma(E) \sim E - E_c$; power-law, $\gamma(E) \sim E^\alpha - E_c^\alpha$; and logarithmic, $\gamma(E) \sim \log(E/E_c)$]. The result is found to be almost independent of the fitting functional and thus provides a reliable estimate of the mobility edge. Figure 2(b) shows the mobility edge versus the quasiperiodic amplitude for various interaction strengths. The error-like bars represent the edges of the minigap containing the mobility edge. The uncertainty on the fitted mobility edge is smaller than these bars.

In order to interpret those results, we now turn to an analytical treatment of the localization problem. The main difficulty relies on the fact that localization in quasiperiodic systems occurs for strong quasiperiodic amplitude Δ [9]. For this reason, the lowest-order perturbation theory, which proved successful for 1D disordered systems [13, 14], fails here [31]. To overcome this issue, we develop an approach based on a generic expansion in harmonics of the quasiperiodic potential. The structure of the GPE (2) shows that the field ϕ_j takes the form of a series of harmonics of the quasiperiodic potential. The density field n_j thus reads $n_j = (\mu - \tilde{V}_j)/U$, where

$$\tilde{V}_j = \sum_{p\geq 1} A_p \cos[p(2\pi r j + \varphi)] \tag{4}$$

is a multi-harmonic quasiperiodic field, the coefficients of which can be computed iteratively [26]. Using the energy-dependent linear transform [14]

$$g_j^{\pm} = \pm\rho_E^{\pm 1/2}(u_j - v_j) + \rho_E^{\mp 1/2}(u_j + v_j) \tag{5}$$

where $\rho_E = \sqrt{1 + (\mu/E)^2} + \mu/E$, the BdGEs (3) exactly rewrite

$$-(\rho_E^{-1}E + \hat{T})g_j^+ + \left[V_j - \frac{3+\rho_E^2}{1+\rho_E^2}\tilde{V}_j\right] g_j^+ = \frac{2\rho_E \tilde{V}_j}{1+\rho_E^2} g_j^- \tag{6}$$

$$(\rho_E E - \hat{T})g_j^- + \left[V_j - \frac{1+3\rho_E^2}{1+\rho_E^2}\tilde{V}_j\right] g_j^- = \frac{2\rho_E \tilde{V}_j}{1+\rho_E^2} g_j^+. \tag{7}$$

The solution of these equations is significantly simplified by noticing that the lattice-space Green function of the operator $-\hat{T} + \rho_E E$ is of width $\sqrt{t/\rho_E E}$ and amplitude $1/(\rho_E E + 2t)\sqrt{1 - [2t/(\rho_E E + 2t)]^2}$. Hence, for $\rho_E E \gg t$, this operator can be replaced by the local operator $\rho_E E + 2t$ in Eq. (7). It is then straightforward to write the expression of g_j^- as a function of g_j^+ and of the potentials V_j and $\tilde{V}_j$. Inserting this expression into Eq. (6) we find a closed equation for g_j^+,

$$-\hat{T} g_j^+ + \mathcal{V}_j^E g_j^+ = E\rho_E^{-1} g_j^+, \tag{8}$$

with the effective potential

$$\mathcal{V}_j^E \simeq V_j - \frac{3+\rho_E^2}{1+\rho_E^2}\tilde{V}_j - \frac{\left(\frac{2\rho_E}{1+\rho_E^2}\right)^2 \tilde{V}_j^2}{\rho_E E + 2t + V_j - \frac{1+3\rho_E^2}{1+\rho_E^2}\tilde{V}_j}. \tag{9}$$

Using exact diagonalization of Eq. (8) with the potential (9) around energy E, we have checked that the Lyapunov exponents and the localization transition given by our effective model coincide with those found using direct diagonalization of the BdGEs (3). It validates the effective model (8)-(9) and the approximation $-\hat{T} + \rho_E E \simeq 2t + \rho_E E$ used above.

In this model, the quantity $\mathcal{V}_j^E$ is a multi-harmonic periodic potential of spacing $1/r$ incommensurate with that of the main lattice, which is unity. Such systems are known to exhibit in general an energy-dependent mobility edge with low-energy extended states and high-energy localized states [27–30]. This holds except in the particular case of self-dual models, among which the Aubry-André model is a celebrated example [9]. Self-duality requires a specific relation between the amplitudes of the p-th harmonics and of the tunelling rate to the p-th neighbors. The latter does not apply in our case since tunelling is strictly restricted to the first neighbors. It qualitatively explains the localization transition of the collective excitations reported here.

Localization properties in quasiperiodic systems can be further inferred from locator perturbation theory [27]. Here the localization criterion roughly corresponds to the convergence of the self-energy in the thermodynamic limit, which reads $D(E) > 1$ where $D(E)$ is the so-called localization function. In the case of Eq. (8), it reads

$$D(E) = \exp\left(r \int_0^{1/r} dx \, \ln \left| \frac{E\rho_E^{-1} - 2t - \mathcal{V}^E(x)}{t} \right| \right). \quad (10)$$

Equation (10) can in principle be applied to the full effective potential $\mathcal{V}^E(x)$. In order to obtain analytical results, it is however worth truncating the infinite series of harmonics in $\mathcal{V}^E$. Keeping only one harmonic is not sufficient to capture the physics even qualitatively, since it would unphysically restore duality and change the universality class of the localization transition. On the other hand, beyond two, the number of harmonics does not change the universality class. We may thus restrict ourselves to the two lowest-order harmonics, which are generated in first instance in second-order perturbation theory. It yields the effective two-harmonic potential

$$\mathcal{V}_j^E \simeq \Delta_E^{(0)} + \Delta_E^{(1)} \cos(2\pi r j + \varphi) + \Delta_E^{(2)} \cos[2(2\pi r j + \varphi)] \quad (11)$$

with the amplitudes

$$\Delta_E^{(0)} = \frac{3+\rho_E^2}{1+\rho_E^2} \frac{\Delta^2}{4Un} (f_r - f_r^2) - \frac{2\rho_E^2}{(1+\rho_E^2)^2} \frac{\Delta^2 f_r^2}{\rho_E E + 2t} \quad (12)$$

$$\Delta_E^{(1)} = \Delta \left[1 - \frac{3+\rho_E^2}{1+\rho_E^2} f_r \right] \quad (13)$$

$$\Delta_E^{(2)} = \frac{3+\rho_E^2}{1+\rho_E^2} \frac{\Delta^2}{4Un} \left[\frac{f_r^2}{2} + \left(f_r - \frac{3}{2} f_r^2 \right) f_{2r} \right] - \frac{2\rho_E^2}{(1+\rho_E^2)^2} \frac{\Delta^2 f_r^2}{\rho_E E + 2t}. \quad (14)$$

and $f_r = \dfrac{1}{1 + 2t \sin^2(\pi r)/Un}$ [26]. The results of the locator perturbation theory applied to the two-harmonic potential (11) is shown on Fig 2(b). It predicts the correct localization transition with collective excitations that are extended at low energy and localized at high energy, and a mobility edge that is in very good agreement with the full numerical result. Perturbation theory beyond second order generates high-order harmonics in the effective potential and renormalizes the amplitudes $\Delta^{(p)}$. For $U \lesssim 3t$, we find that they induce negligible effects and do not significantly affect the prediction for the mobility edge. For higher values of U, however, second-order perturbation theory is not sufficient to accurately estimate the density background, and higher-order terms should be included.

In summary, we have shown that the collective excitations of lattice Bose superfluids subjected to a single-harmonic quasiperiodic potential undergo a localization transition with extended low-energy states and localized high-energy states. Therefore the interactions change the universality class of the localization transition. This is in striking contrast with the purely disordered case [13, 14]. In the quasiperiodic case the transition can be understood as the result of the scattering of the excitations from the potential and the density background, which contains an infinite series of harmonics of the potential. It could be observed in ultracold-atom experiments, using for instance spectroscopy techniques, which give direct access to the excitations [32]. Alternatively, it could be observed in quench dynamics, where the abrupt change of some physical parameter generates collective excitations that govern the propagation of experimentally-observable correlations [20, 33].

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Supplemental Material

This supplemental material aims at providing some details about the harmonic structure of the density profile and the effective potential $\mathcal{V}_j^E$, as well as the series expansion used to determine them.

Series expansion of the density background

To determine the density background $n_j \equiv \phi_j^2$, we solve the Gross-Pitaevskii equation (GPE) together with the normalization condition $n = \frac{1}{L}\Sigma_j n_j$ where L is the number of sites and n is the averaged density. To do so, we perform a series expansion in powers of the quasiperiodic potential V_j. In the absence of an external potential, we have $\phi_j = \sqrt{n}$ and the chemical potential $\mu = Un$. In the presence of an external potential, we then write

$$\phi_j = \sqrt{n}\left(\phi_j^{(0)} + \phi_j^{(1)} + \phi_j^{(2)}...\right) \tag{S1}$$

$$\mu = Un\left(\mu^{(0)} + \mu^{(1)} + \mu^{(2)} + ...\right) \tag{S2}$$

$$n_j = n\left(n_j^{(0)} + n_j^{(1)} + n_j^{(2)}...\right) \tag{S3}$$

where the superscripts denote increasing orders in the quasiperiodic amplitude Δ, and $\phi_j^{(0)} = 1$, $n_j^{(0)} = 1$, $\mu^{(0)} = 1$. Notice that the chemical potential has to be expanded also to fulfill the normalization condition. Inserting the expansions (S1) and (S2) in the GPE [Eq. (2) of the paper] and the expansion (S3) in the normalization condition, we get

$$Un\left(\mu^{(0)} + \mu^{(1)} + ...\right)\left(\phi_j^{(0)} + \phi_j^{(1)} + ...\right) = -\hat{T}\left(\phi_j^{(0)} + \phi_j^{(1)} + ...\right) + V_j\left(\phi_j^{(0)} + \phi_j^{(1)} + ...\right) + Un\left(\phi_j^{(0)} + \phi_j^{(1)} + ...\right)^3 \tag{S4}$$

and

$$\frac{1}{L}\sum_j \left(\phi_j^{(0)} + \phi_j^{(1)} + ...\right)^2 = 1. \tag{S5}$$

Then, collecting all the terms of same order p in the quasiperiodic amplitude yields

$$\left(1 - \frac{1}{2Un}\hat{T}\right)\phi_j^{(p)} = -\frac{V_j}{2Un}\phi_j^{(p-1)} - \frac{1}{2}\sum_{k,\ell,m=p,0\leq k,\ell,m\leq p-1} \phi_j^{(k)}\phi_j^{(\ell)}\phi_j^{(m)} + \frac{1}{2}\sum_{1\leq k\leq p-1} \mu^{(k)}\phi_j^{(p-k)} + \frac{\mu^{(p)}}{2} \tag{S6}$$

and

$$\sum_j \left(2\phi_j^{(0)}\phi_j^{(p)} + \sum_{1\leq k\leq p-1} \phi_j^{(k)}\phi_j^{(p-k)}\right) = 0. \tag{S7}$$

Equations (S6) and (S7) can then be used to compute all $\phi_j^{(p)}$ and $\mu^{(p)}$ at any order p iteratively. The iteration process works as follows. Given all $\phi_j^{(k)}$ and $\mu^{(k)}$ at orders $k < p$, we calculate $\phi_j^{(p)}$ as a function of $\mu^{(p)}$ from Eq. (S6) by inverting the operator $1 - \hat{T}/2Un$. The quantity $\mu^{(p)}$ is then found by inserting this expression for $\phi_j^{(p)}$ into Eq. (S7). Having determined $\phi_j^{(p)}$, we then find the density field using Eq. (S3), whose expansion in powers of the quasiperiodic amplitude writes

$$n_j^{(p)} = \sum_{0\leq k,\ell\leq p, k+\ell=p} \phi_j^{(k)}\phi_j^{(\ell)}. \tag{S8}$$

This procedure is completely general and can be applied to any external potential V_j. In the case of the quasiperiodic potential $V_j = \Delta\cos(2\pi r j + \varphi)$, the above iterative process is fully algebraic because the operator $1 - \hat{T}/2Un$ in Eq. (S6) can be analytically inverted at any order (see below). We have implemented this expansion up to order 50 and found excellent agreement with the direct numerical solution of the GPE (2). It provides a cross-check of the precision of the numerical solution and of the convergence of the present analytical expansion.

Analytical expansion in the case of a quasiperiodic potential

We now give some explicit formulas for the lowest order terms and discuss the harmonic structure of the density background n_j. As in the paper, we generically write the density field n_j

$$n_j = (\mu - \tilde{V}_j)/U, \tag{S9}$$

where the field $\tilde{V}$ includes terms of all orders.

At first order, Eq. (S6) reduces to $-\hat{T}\phi_j^{(1)} + 2Un\phi_j^{(1)} = -V_j + Un\mu^{(1)}$. It is straightforward to solve it in Fourier space where the operator $\hat{T}$ is diagonal. It yields $\phi_k^{(1)} = -\dfrac{V_k - Un\mu^{(1)}\delta_{k,0}}{\varepsilon_k^0 + 2Un}$ where $\varepsilon_k^0 = 4t\sin^2(k/2)$. Inserting this expression into Eq. (S7), we find $\mu^{(1)} = V_{k=0}/Un = 0$ and $\phi_k^{(1)} = -\dfrac{V_k}{\varepsilon_k^0 + 2Un}$. Remarkably, since the quasiperiodic potential contains only one spatial frequency, $V_k = \Delta(e^{i\varphi}\delta_{k,+2\pi r} + e^{-i\varphi}\delta_{k,-2\pi r})/2$, one can immediately get back to real space and write

$$\phi_j^{(1)} = -\frac{\Delta}{2Un} f_r \cos(2\pi r j + \varphi), \tag{S10}$$

where $f_r = \dfrac{1}{1 + \varepsilon_{2\pi r}^0/2Un}$. Hence, to lowest order, the density profile is quasiperiodic field. It follows the modulations of the quasiperiodic potential with a reduced amplitude since $f_r < 1$. The factor f_r is a remainder of the nonlocal operator $1 - \hat{T}/2Un$ in the l.h.s. of Eq. (S6), which reduces to an algebraic operation in the case of a quasiperiodic potential. Then, Eqs. (S8) and (S9) yield the first order term of the field $\tilde{V}_j$. It reads $\tilde{V}_j^{(1)} = -2Un\phi_j^{(1)}$ where $\phi_j^{(1)}$ is given by Eq. (S10), i.e.

$$\tilde{V}_j^{(1)} = \Delta f_r \cos(2\pi r j + \varphi). \tag{S11}$$

The next orders are found following the same process, which remains algebraic to any order in the case of the quasiperiodic potential. To second order, it yields the term

$$\phi_j^{(2)} = \left(\frac{\Delta}{2Un}\right)^2 \left[-\frac{f_r^2}{4} + \left(f_r - \frac{3}{2}f_r^2\right) f_{2r} \frac{\cos[2(2\pi r j + \varphi)]}{2} \right], \tag{S12}$$

and a negative shift on the chemical potential,

$$\mu^{(2)} = -\left(\frac{\Delta}{2Un}\right)^2 (f_r - f_r^2). \tag{S13}$$

The field $\tilde{V}_j$ is then given at second order by $\tilde{V}_j^{(2)} = Un[\mu^{(2)} - n_j^{(2)}] = Un[\mu^{(2)} - (2\phi_j^{(2)} + \phi_j^{(1)2})]$ where $\phi^{(1)}$, $\phi^{(2)}$ and $\mu^{(2)}$ are given by Eqs. (S10), (S12), and (S13), i.e.

$$\tilde{V}_j^{(2)} = -\frac{\Delta^2}{4Un} \left\{ f_r - f_r^2 + \left[\frac{f_r^2}{2} + \left(f_r - \frac{3}{2}f_r^2\right) f_{2r} \right] \cos[2(2\pi r j + \varphi)] \right\}. \tag{S14}$$

Hence, the second-order terms $\phi_j^{(2)}$ and $\tilde{V}_j^{(2)}$ contain a constant term and the second harmonics of the quasiperiodic potential. Those terms are generated by the nonlinear term of the GPE: Since the first order term contains only the first harmonics, $\phi_j^{(1)} \propto \cos(2\pi r j + \varphi)$, the product terms $\phi_j^{(1)}\phi_j^{(1)}\phi_j^{(0)}$ appearing in Eq. (S6) contain the zeroth and second harmonics.

More generally, it is straightforward to show recursively that the terms of order p, $\phi_j^{(p)}$ and $\tilde{V}_j^{(p)}$, contain the p-th harmonics of the quasiperiodic potential, $\cos[p(2\pi r j + \varphi)]$, as well as all lower harmonics of same parity. In particular, a constant term in ϕ_j and a correction to the chemical potential μ appear only at even orders. Hence, the field $\tilde{V}_j$ takes the multi-harmonic quasiperiodic form

$$\tilde{V}_j = \sum_p A_p \cos[p(2\pi r j + \varphi)], \tag{S15}$$

where the amplitude A_p of the p-th harmonics is a power series of order p, $A_p \sim \alpha_p(\Delta/2Un)^p + \alpha_{p+2}(\Delta/2Un)^{p+2} + ...$

Expansion of the effective potential $\mathcal{V}_j^E$

The effective potential

$$\mathcal{V}_j^E \simeq V_j - \frac{3+\rho_E^2}{1+\rho_E^2}\tilde{V}_j - \frac{\left(\frac{2\rho_E}{1+\rho_E^2}\right)^2 \tilde{V}_j^2}{\rho_E E + 2t + V_j - \frac{1+3\rho_E^2}{1+\rho_E^2}\tilde{V}_j}, \tag{S16}$$

which appears in the Eq. (8) of the paper, can as well be expanded in powers of the quasiperiodic amplitude by expanding the denominator and using the previously obtained expansions for $\tilde{V}_j$. Up to second order, we get

$$\mathcal{V}_j^E \simeq V_j - \frac{3+\rho_E^2}{1+\rho_E^2}\tilde{V}_j^{(1)} - \frac{3+\rho_E^2}{1+\rho_E^2}\tilde{V}_j^{(2)} - \frac{\left(\frac{2\rho_E}{1+\rho_E^2}\right)^2}{\rho_E E + 2t}\left(\tilde{V}_j^{(1)}\right)^2, \tag{S17}$$

where $\tilde{V}^{(1)}$ and $\tilde{V}^{(2)}$ are given in Eqs. (S11) and (S14). It yields the two-harmonic effective potential

$$\mathcal{V}_j^E \simeq \Delta_E^{(0)} + \Delta_E^{(1)}\cos(2\pi r j + \varphi) + \Delta_E^{(2)}\cos[2(2\pi r j + \varphi)] \tag{S18}$$

with the amplitudes

$$\Delta_E^{(1)} = \Delta\left[1 - \frac{3+\rho_E^2}{1+\rho_E^2} f_r\right] \tag{S19}$$

$$\Delta_E^{(2)} = \frac{3+\rho_E^2}{1+\rho_E^2}\frac{\Delta^2}{4Un}\left[\frac{f_r^2}{2} + \left(f_r - \frac{3}{2}f_r^2\right) f_{2r}\right] - \frac{2\rho_E^2}{(1+\rho_E^2)^2}\frac{\Delta^2 f_r^2}{\rho_E E + 2t} \tag{S20}$$

$$\Delta_E^{(0)} = \frac{3+\rho_E^2}{1+\rho_E^2}\frac{\Delta^2}{4Un}(f_r - f_r^2) - \frac{2\rho_E^2}{(1+\rho_E^2)^2}\frac{\Delta^2 f_r^2}{\rho_E E + 2t}. \tag{S21}$$