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We study the competition between one-body and two-body couplings in weakly-interacting two-component Bose gases, in particular as regards field correlations. We derive the meanfield theory for both ground state and low-energy pair excitations in the general case where both one-body and two-body couplings are position-dependent and the fluid is subjected to a state-dependent trapping potential. General formulas for phase and density correlations are also derived. Focusing on the case of homogeneous systems, we discuss the pair-excitation spectrum and the corresponding excitation modes, and use them to calculate correlation functions, including both quantum and thermal fluctuation terms. We show that the relative phase of the two components is imposed by that of the one-body coupling, while its fluctuations are determined by the modulus of the one-body coupling and by the two-body coupling. One-body coupling and repulsive two-body coupling cooperate to suppress relative-phase fluctuations, while attractive two-body coupling tends to enhance them. Further applications of the formalism presented here and extensions of our work are also discussed.

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I. INTRODUCTION

Multi-component (spinor) quantum fluids underlie a variety of physical systems, such as ^{3}He-^{4}He mixtures in three-fluid models [1], Bose-condensed spin-polarized Hydrogen gases in the two lowest-energy states [2–4], optically-excited excitons in high-quality $Cu_2 0$ crystals [5, 6], as well as gaseous Bose-Einstein condensates either in two overlaped atomic hyperfine states [7–9] or in adjacent traps coupled by tunnel effect [10]. The dynamics of spinors sparks a variety of physical effects, including quantum phase transitions, topological defects, and spin domains, governed by the complex interplay of particle-particle interactions, exchange coupling, magnetic-like ordering, and temperature effects. Early studies focused on the possibility of observing Bose-Einstein condensation [11], as well as stability conditions [1, 12, 13], phase separation [8, 14–20], and spontaneous symmetry breaking mechanisms [21–24] in two-component Bose-Einstein condensates. Two-component Bose gases have also been used to study phase coherence [25], Josephson like physics [26–30], the dynamics of spin textures [31–34], random-field-induced order effects [35, 36], and twin quantum states for quantum information processing [37–39].

In the context of ultracold gases the combination of optical and magnetic fields designed to manipulate the internal states of alkali atoms offer a wide range of possibilities to accurately engineer multi-component quantum fluids. Such systems offer a new tool to study quantum coherence in various contexts [9, 25, 27, 30]. For instance, measurement of the relative-phase correlation function of a coupled binary Bose gas in one dimension was reported in Ref. [30]. In the later case, the coupling was of the Josephson (one-body) type.

In this paper, we consider a two-component Bose gas with both one-body (field-field) and two-body (density-density) couplings, and focus our analysis on the pair-excitation spectrum and the relative phase correlation function at both zero and finite temperature. The most general case can be realized in ultracold-atom gases by using a mixture of atoms in two different internal hyperfine states (noted 1 and 2) of the same atomic species. The two-body interaction with coupling constant g_{12} results from short-range particle-particle interactions between atoms in different internal states, while the one-body interaction can be implemented by two-photon Raman optical coupling, which transfers atoms from one internal state to the other (see schematic view on Fig. 1). In Sec. II, we present the model and derive the mean-field theory of the coupled two-component Bose fluid for both ground state and low-energy pair excitations.

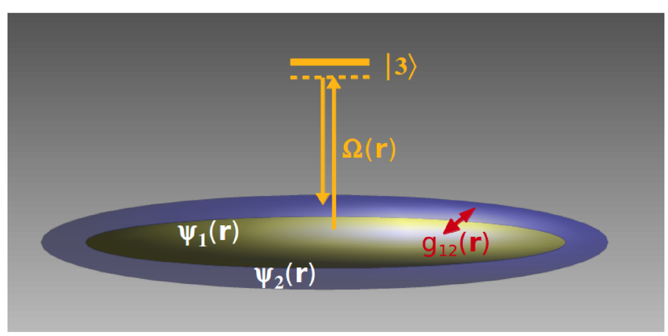

FIG. 1. Coupled two-component Bose gas. The gas is made of bosonic particles of a single atomic species, which can be in two different internal states (labeled 1 and 2). It is described by the two field operators $\hat{\psi}_1(\mathbf{r})$ and $\hat{\psi}_2(\mathbf{r})$, corresponding to each component. In this work, we assume that the two components are coupled by one-body and/or two-body interactions of coupling constants Ω and g_{12}, respectively. In the most general case, the two coupling constants can be position dependent.

The theory is formulated in the most general case, where both one-body and two-body couplings are position dependent and the fluid is subjected to a state-dependent trapping potential. In addition, we use the phase-density Bogoliubov-Popov approach, which allows us to treat true condensates and quasi-condensates on equal footing [40, 41]. General formulas for phase and density correlations are derived. In Sec. III, we focus on the case of homogeneous systems, which allow considerable simplification of the formalism and contain most of the physical effects. After rewriting the general meanfield equations for homogeneous systems (III A), we discuss the pair-excitation spectrum and the corresponding fields, and use them to calculate the correlation functions including both quantum and thermal fluctuation terms. We distinguish three cases: (i) two-body coupling alone (Sec. III B), (ii) one-body coupling alone (Sec. III C), and (iii) both one-body and two-body couplings (Sec. III D). The analysis of these cases leads to the following conclusions: The phase of the one-body coupling term imposes alone the relative phase of the two components at the meanfield background level. Then, the fluctuations of the relative phase are determined by the interplay of the modulus of the one-body term and by the two-body term. On the one hand, the one-body coupling always favors local mutual coherence of the two components but the correlation length decreases when the modulus of the one-body term increases. On the other hand, repulsive two-body coupling cooperates with one-body coupling to further suppress relative-phase fluctuations, while attractive two-body coupling competes with one-body coupling to enhance relative-phase fluctuations. These results are summarized in more detail in Sec. IV, where we also discuss further possible applications of the formalism presented here.

II. MEAN-FIELD THEORY OF A TWO-COMPONENT BOSE GAS

Consider a two-component Bose-Bose mixture at thermodynamic equilibrium at temperature T, and in the weakly interacting regime. We assume that the two components (labelled by $\sigma \in \{1, 2\}$) interact with each other and can exchange atoms to maintain chemical equilibrium. The average total number of atoms, $N = N_1 + N_2$, is conserved but the average number of atoms in each component, N_σ, is not. The physics of this system is governed by the grand-canonical Hamiltonian

$$\hat{H} \equiv \hat{\mathcal{H}} - \mu \hat{N} = \hat{H}_1 + \hat{H}_2 + \hat{H}_{12}\,, \tag{1}$$

where $\hat{\mathcal{H}}$ is the many-body Hamiltonian and $\hat{N} = \hat{N}_1 + \hat{N}_2$ is the total number operator, with $\hat{N}_\sigma = \int \mathrm{d}\mathbf{r}\; \hat{\psi}_\sigma^\dagger(\mathbf{r})\hat{\psi}_\sigma(\mathbf{r})$ and $\hat{\psi}_\sigma(\mathbf{r})$ the (bosonic) field operator of component σ. Assuming two-body contact interactions, the Hamiltonian associated the the sole component σ (written in the grand-canonical form for the chemical potential μ of the mixture) is

$$\hat{H}_\sigma = \int \mathrm{d}\mathbf{r}\; \hat{\psi}_\sigma^\dagger \left[-\frac{\hbar^2\nabla^2}{2m} + V_\sigma - \mu + \frac{g_\sigma(\mathbf{r})}{2}\hat{\psi}_\sigma^\dagger\hat{\psi}_\sigma \right] \hat{\psi}_\sigma \tag{2}$$

and the coupling Hamiltonian is

$$\hat{H}_{12} = \int \mathrm{d}\mathbf{r} \left[g_{12}(\mathbf{r})\hat{\psi}_1^\dagger\hat{\psi}_2^\dagger\hat{\psi}_1\hat{\psi}_2 + \left(\frac{\hbar\Omega(\mathbf{r})}{2}\hat{\psi}_2^\dagger\hat{\psi}_1 + \mathrm{H.c.} \right) \right]. \tag{3}$$

The single-component Hamiltonian $\hat{H}_\sigma$ contains (i) a kinetic term (m is the atomic mass), (ii) a potential term, $V_\sigma(\mathbf{r})$, both associated with single-particle dynamics, and (iii) an intra-component interaction term of coupling parameter g_σ. The coupling Hamiltonian, $\hat{H}_{12}$, contains (i) a term originating from elastic contact interaction between two atoms in different components characterized by the inter-component coupling constant g_{12}, and (ii) an exchange term proportional to Ω, which transfers atoms from one component to the other and in particular permits chemical equilibrium. In ultracold-atom systems, the exchange one-body term can be realized by two-photon Raman or radio-frequency coupling [7] or by Josephson coupling between two adjacent traps [26, 30, 42–44], whereas the two-body coupling can be controlled by Feshbach resonance techniques [45]. In the most general case, all coupling terms g_1, g_2, g_{12}, and Ω can be position-dependent. Hereafter, we write $\Omega(\mathbf{r}) \equiv \Omega_0(\mathbf{r})\mathrm{e}^{-i\alpha(\mathbf{r})}$, with $\Omega_0 = |\Omega|$ and $\alpha(\mathbf{r})$ the phase of the exchange coupling, for convenience.

In the following, we first reformulate the above Hamiltonians into the phase-density formalism, which is more appropriate for our study. We then apply the Gross-Pitaevskii approach, which describes the meanfield quasi-condensate background of the two-component Bose-Bose mixture, and develop the Bogoliubov-de Gennes theory for the mixture, which provides the spectrum of collective excitations and can be used to describe finite-temperature effects. We finally write down the general expressions for the density and phase correlation functions, which will be calculated in the next sections. Although the process we follow is standard, we generalize previous work to the case where their couplings can be position-dependent. We thus detail the derivation of the main equations.

A. Phase-density formalism

The complete grand-canonical Hamiltonian $\hat{H}$ is invariant under the gauge transformation $\{\hat{\psi}_1(\mathbf{r}), \hat{\psi}_2(\mathbf{r})\} \to \mathrm{e}^{i\theta_0}\{\hat{\psi}_1(\mathbf{r}), \hat{\psi}_2(\mathbf{r})\}$ for any value of $\theta_0 \in \mathbb{R}$, as can be easily checked in Eqs. (2) and (3). More precisely, if $\Omega(\mathbf{r}) \equiv 0$, the phases of the two components are independent and $\hat{H}$ is invariant under the more general transformation $\{\hat{\psi}_1(\mathbf{r}), \hat{\psi}_2(\mathbf{r})\} \to \{\mathrm{e}^{i\theta_0^1}\hat{\psi}_1(\mathbf{r}), \mathrm{e}^{i\theta_0^2}\hat{\psi}_2(\mathbf{r})\}$ for any values of $\theta_0^1, \theta_0^2 \in \mathbb{R}$. If however $\Omega(\mathbf{r}) \not\equiv 0$, the phases of the two components are coupled via the last term in Eq. (3) and the relative phase is a determined quantity. In both cases, the phases of the field operators $\hat{\psi}_\sigma(\mathbf{r})$ are not fully determined and it is useful to turn to the phase-density formalism. We write the field operator for each component in the form

$$\hat{\psi}_\sigma(\mathbf{r}) = \mathrm{e}^{i\hat{\theta}_\sigma(\mathbf{r})}\sqrt{\hat{n}_\sigma(\mathbf{r})}, \tag{4}$$

where the density ($\hat{n}_\sigma$) and phase ($\hat{\theta}_\sigma$) operators satisfy the Bose commutation rule $[\hat{n}_\sigma(\mathbf{r}), \hat{\theta}_{\sigma'}(\mathbf{r}')] = i\delta_{\sigma\sigma'}\delta(\mathbf{r}-\mathbf{r}')$. Replacing $\hat{\psi}_\sigma$ by expression (4) into Eqs. (2) and (3), we find

$$\hat{H}_\sigma = \int \mathrm{d}\mathbf{r}\,\sqrt{\hat{n}_\sigma}\left[\frac{-\hbar^2}{2m}\left(\nabla^2 - |\nabla\hat{\theta}_\sigma|^2\right) + V_\sigma - \mu + \frac{g_\sigma}{2}\hat{n}_\sigma\right]\sqrt{\hat{n}_\sigma} \tag{5}$$

and

$$\hat{H}_{12} = \int \mathrm{d}\mathbf{r}\,\left[g_{12}\hat{n}_1\hat{n}_2 + \left\{\frac{\hbar\Omega}{2}\sqrt{\hat{n}_2}\mathrm{e}^{i(\hat{\theta}_1-\hat{\theta}_2)}\sqrt{\hat{n}_1} + \mathrm{H.c.}\right\}\right]. \tag{6}$$

Expressions (5) and (6) determine the complete Hamiltonian (1) in terms of density and phase operators [81]. This form is particularly suitable for perturbative expansion in the condensate or quasi-condensate regime, where the density fluctuations are suppressed by strong-enough repulsive interactions but the phase fluctuations can be large [40, 41, 47–49].

B. Meanfield background: Gross-Pitaevskii theory

The zeroth-order term in quantum and thermal fluctuations corresponds to the meanfield background. The latter is determined using the Gross-Pitaevskii approach [50, 51], adapted to the two-component mixture. It amounts to minimize the grand-canonical energy functional $E_{\mathrm{MF}} \equiv \langle\psi_{\mathrm{MF}}|\hat{H}|\psi_{\mathrm{MF}}\rangle$ with the two-component Hartree-Fock ansatz

$$|\psi_{\mathrm{MF}}\rangle = \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}}\,\frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}}\,|\mathrm{vac}\rangle\,, \tag{7}$$

where $\hat{a}_\sigma^\dagger$ creates an atom in component σ with a spatial wave function $\psi_\sigma(\mathbf{r}) \equiv \mathrm{e}^{i\theta_\sigma(\mathbf{r})}\sqrt{n_\sigma(\mathbf{r})}$. At this stage, the number of atoms in each component, N_σ, and the corresponding phase $[\theta_\sigma(\mathbf{r})]$ and density $[n_\sigma(\mathbf{r})]$ fields are unknown variational quantities. Here, we use the normalization condition $\int \mathrm{d}\mathbf{r}\; n_\sigma(\mathbf{r}) = N_\sigma$ and we recall that the chemical potential μ is determined implicitly by the relation $\int \mathrm{d}\mathbf{r}\;[n_1(\mathbf{r}) + n_2(\mathbf{r})] = N$.

Proceeding in the standard way, we evaluate the complete grand-canonical Hamiltonian (1) within the Hartree-Fock ansatz (7) and find

$$E_{\mathrm{MF}} = \langle\hat{H}_1\rangle_{\mathrm{MF}} + \langle\hat{H}_2\rangle_{\mathrm{MF}} + \langle\hat{H}_{12}\rangle_{\mathrm{MF}} \tag{8}$$

where $\langle\hat{H}_\sigma\rangle_{\mathrm{MF}}$ and $\langle\hat{H}_{12}\rangle_{\mathrm{MF}}$ are given by Eqs. (5) and (6) with the phase $\hat{\theta}_\sigma(\mathbf{r})$ and density $\hat{n}_\sigma(\mathbf{r})$ operators replaced by the corresponding Hartree-Fock fields $\theta_\sigma(\mathbf{r})$ and $n_\sigma(\mathbf{r})$. Then, minimizing E_{MF} with respect to $\theta_\sigma(\mathbf{r})$ and $n_\sigma(\mathbf{r})$ yields the following coupled Euler-Lagrange equations:

$$0 = -\frac{\hbar^2}{2m}\left(\frac{\nabla^2\sqrt{n_\sigma}}{\sqrt{n_\sigma}} - |\nabla\theta_\sigma|^2\right) + V_\sigma - \mu + g_\sigma n_\sigma + g_{12}n_{\bar{\sigma}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar{\sigma}}}{n_\sigma}}\cos(\theta-\alpha) \tag{9}$$

$$0 = \frac{\hbar^2}{m}\nabla(n_\sigma\nabla\theta_\sigma) \pm \hbar\Omega_0\sqrt{n_1 n_2}\sin(\theta-\alpha)\,, \tag{10}$$

where $\theta(\mathbf{r}) \equiv \theta_1(\mathbf{r}) - \theta_2(\mathbf{r})$ is the relative phase between the two components, $\bar{\sigma}$ is the conjugate of σ [i.e. $\bar{\sigma} = 2$ (resp. 1) for $\sigma = 1$ (resp. 2)], and the $\pm$ sign in Eq. (10) is $+$ (resp. $-$) for $\sigma = 1$ (resp. 2).

C. Excitations: Bogoliubov-de Gennes theory

The low-energy spectrum of the collective excitations of the two-component Bose gas is then determined using the Bogoliubov-de Gennes approach [40, 41, 52–54], which amounts to perform a perturbative expansion of Hamiltonian (1) in phase and density fluctuations. We write $\hat{n}_\sigma = n_\sigma + \delta\hat{n}_\sigma$ and $\hat{\theta}_\sigma = \theta_\sigma + \delta\hat{\theta}_\sigma$, with $n_\sigma(\mathbf{r})$ and $\theta_\sigma(\mathbf{r})$ given by the mean-field Gross-Pitaevskii theory, and

$$|\delta\hat{n}_\sigma| \ll n_\sigma \quad \text{and} \quad |\nabla\delta\hat{\theta}_\sigma| \ll mc/\hbar \tag{11}$$

where $c = \sqrt{\mu/m}$ is the velocity of sound in a single-component Bose-Einstein (quasi-)condensate of chemical potential μ. These conditions are usually well verified in weakly-interacting ultracold, two-component gases [7–9, 55].

1. *Weak-fluctuation expansion of the Hamiltonian*

Proceeding up to second order in phase and density fluctuations, it is convenient to define the position-dependent operators

$$\hat{X}_\sigma(\mathbf{r}) \equiv \frac{\delta\hat{n}_\sigma(\mathbf{r})}{2\sqrt{n_\sigma(\mathbf{r})}} \tag{12}$$

and

$$\hat{P}_\sigma(\mathbf{r}) \equiv \sqrt{n_\sigma(\mathbf{r})}\delta\hat{\theta}_\sigma(\mathbf{r})\,, \tag{13}$$

which are canonical conjugates (up to a multiplying factor of 1/2), i.e. $\left[\hat{X}_\sigma(\mathbf{r}), \hat{P}_{\sigma'}(\mathbf{r}')\right] = i\delta_{\sigma,\sigma'}\delta\left(\mathbf{r}-\mathbf{r}'\right)/2$. Then, inserting $\sqrt{\hat{n}_\sigma} \simeq \sqrt{n_\sigma} + \hat{X}_\sigma - \hat{X}_\sigma^2/2\sqrt{n_\sigma}$ and $\hat{\theta}_\sigma = \theta_\sigma + \hat{P}/\sqrt{n_\sigma}$ into Eqs. (5) and (6), we find

$$\hat{H} \simeq E_{\mathrm{MF}} + \hat{H}_1^{(2)} + \hat{H}_2^{(2)} + \hat{H}_{12}^{(2)}\,. \tag{14}$$

The zeroth-order term, E_{MF}, coincides with the mean-field energy (8) where the fields n_σ and θ_σ are substituted to the solutions of the coupled Euler-Lagrange equations (9) and (10). The first-order term, $\hat{H}^{(1)} = \sum_\sigma \left\{ \delta\hat{n}_\sigma \cdot \frac{\partial \hat{H}}{\partial \hat{n}_\sigma}\Big|_{\psi_{\text{MF}}} + \delta\hat{\theta}_\sigma \cdot \frac{\partial \hat{H}}{\partial \hat{\theta}_\sigma}\Big|_{\psi_{\text{MF}}} \right\}$, vanishes since the zeroth-order term minimizes $\langle\psi_{\text{MF}}|\hat{H}|\psi_{\text{MF}}\rangle = E_{\text{MF}}$. The second-order terms, $\hat{H}_1^{(2)}, \hat{H}_2^{(2)}$ and $\hat{H}_{12}^{(2)}$, are found after some straightforward algebra, which yields

$$\begin{aligned}\hat{H}_\sigma^{(2)} = &\int \mathrm{d}\mathbf{r}\ \hat{X}_\sigma \left[-\frac{\hbar^2}{2m}\left(\nabla^2 - \frac{\nabla^2\sqrt{n_\sigma}}{\sqrt{n_\sigma}}\right) + 2g_\sigma n_\sigma \right] \hat{X}_\sigma \\ &+ \int \mathrm{d}\mathbf{r}\ \hat{P}_\sigma \left[-\frac{\hbar^2}{2m}\left(\nabla^2 - \frac{\nabla^2\sqrt{n_\sigma}}{\sqrt{n_\sigma}}\right)\right] \hat{P}_\sigma \\ &+ \int \mathrm{d}\mathbf{r}\ \frac{2\hbar^2}{m}\nabla\theta_\sigma \cdot \left(\sqrt{n_\sigma}\hat{X}_\sigma\right)\nabla\left(\hat{P}_\sigma/\sqrt{n_\sigma}\right),\end{aligned} \tag{15}$$

where some irrelevant constant terms have been dropped, and

$$\begin{aligned}\hat{H}_{12}^{(2)} = &-\sum_\sigma \int \mathrm{d}\mathbf{r}\ \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}}\cos(\theta-\alpha)\left[\hat{X}_\sigma^2 + \hat{P}_\sigma^2\right] \\ &+ \int \mathrm{d}\mathbf{r}\ \left[4g_{12}\sqrt{n_1 n_2} + \hbar\Omega_0\cos(\theta-\alpha)\right]\hat{X}_1\hat{X}_2 \\ &+ \int \mathrm{d}\mathbf{r}\ \hbar\Omega_0\cos(\theta-\alpha)\hat{P}_1\hat{P}_2 \\ &+ \int \mathrm{d}\mathbf{r}\ \hbar\Omega_0\sin(\theta-\alpha)\left[\hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1\right] \\ &- \int \mathrm{d}\mathbf{r}\ \hbar\Omega_0\sin(\theta-\alpha)\left[\frac{\sqrt{n_2}}{\sqrt{n_1}}\hat{X}_1\hat{P}_1 - \frac{\sqrt{n_1}}{\sqrt{n_2}}\hat{X}_2\hat{P}_2\right].\end{aligned} \tag{16}$$

We now apply the canonical transformation [82] to our quadratic Hamiltonian [83]

$$\hat{B}_\sigma \equiv \hat{X}_\sigma + i\hat{P}_\sigma\,, \tag{17}$$

such that the operators $\hat{B}_\sigma$ satisfy the Bose commutation rules

$$[\hat{B}_\sigma(\mathbf{r}), \hat{B}_{\sigma'}(\mathbf{r}')] = 0 \tag{18}$$

$$[\hat{B}_\sigma(\mathbf{r}), \hat{B}^\dagger_{\sigma'}(\mathbf{r}')] = \delta_{\sigma\sigma'}\delta(\mathbf{r}-\mathbf{r}')\,. \tag{19}$$

Then, summing all contributions of Eq. (15) for $\sigma = 1$ and $\sigma = 2$ and those of Eq. (16), we find

$$\begin{aligned}\hat{H}^{(2)} = &\frac{1}{2}\sum_\sigma \int \mathrm{d}\mathbf{r}\ \Big[\hat{B}_\sigma^\dagger \mathbf{A}_\sigma \hat{B}_\sigma + \hat{B}_\sigma \mathbf{A}_\sigma^* \hat{B}_\sigma^\dagger \\ &\qquad + \left\{ g_\sigma n_\sigma \hat{B}_\sigma \hat{B}_\sigma + \text{H.c.}\right\}\Big] \\ &+ \int \mathrm{d}\mathbf{r}\ \left[g_{12}\sqrt{n_1 n_2}\hat{B}_1\hat{B}_2 + \text{H.c.}\right] \\ &+ \int \mathrm{d}\mathbf{r}\ \left[\left\{g_{12}\sqrt{n_1 n_2} + \frac{\hbar\Omega}{2}e^{i\theta}\right\}\hat{B}_2^\dagger\hat{B}_1 + \text{H.c.}\right]\end{aligned} \tag{20}$$

where we have used the coupled Euler-Lagrange equation (9) to simplify a couple a terms, and have introduced the super-operator

$$\begin{aligned}\mathbf{A}_\sigma = &-\frac{\hbar^2}{2m}\left(\nabla^2 + 2i\nabla\theta_\sigma\cdot\nabla - |\nabla\theta_\sigma|^2\right) + V_\sigma - \mu \\ &+ 2g_\sigma n_\sigma + g_{12} n_{\bar\sigma}\,.\end{aligned} \tag{21}$$

Finally, the Hamiltonian (20) can be written in a more compact form by introducing the four-component operators

$$\bar{\mathcal{B}} \equiv \left[\hat{B}_1^\dagger, -\hat{B}_1, \hat{B}_2^\dagger, -\hat{B}_2\right] \quad \text{and} \quad \mathcal{B} \equiv \begin{bmatrix}\hat{B}_1 \\ \hat{B}_1^\dagger \\ \hat{B}_2 \\ \hat{B}_2^\dagger\end{bmatrix} \tag{22}$$

so that

$$\hat{H}^{(2)} = \frac{1}{2}\int \mathrm{d}\mathbf{r}\ \bar{\mathcal{B}}(\mathbf{r})\mathbf{M}(\mathbf{r})\mathcal{B}(\mathbf{r}) + \text{const} \tag{23}$$

where $\mathbf{M}(\mathbf{r})$ is the 4×4 super-operator defined by

$$\mathbf{M} \equiv \begin{bmatrix} \mathcal{L}_1^{\text{GP}} & \Gamma \\ \Gamma^* & \mathcal{L}_2^{\text{GP}} \end{bmatrix} \tag{24}$$

with

$$\mathcal{L}_\sigma^{\text{GP}} = \begin{bmatrix} +\mathbf{A}_\sigma & +g_\sigma n_\sigma \\ -g_\sigma n_\sigma & -\mathbf{A}_\sigma^* \end{bmatrix} \tag{25}$$

and

$$\Gamma = \begin{bmatrix} +g_{12}\sqrt{n_1 n_2} + \frac{\hbar\Omega^*}{2}e^{-i\theta} & +g_{12}\sqrt{n_1 n_2} \\ -g_{12}\sqrt{n_1 n_2} & -g_{12}\sqrt{n_1 n_2} - \frac{\hbar\Omega}{2}e^{+i\theta} \end{bmatrix}. \tag{26}$$

2. *Bogoliubov transformation*

The second-order term (23) in the expansion of the many-body Hamiltonian (1) governs the low-energy excitations of the two-component Bose gas. Its quadratic form is convenient for diagonization via the usual Bogoliubov method [40, 41, 52, 53], adapted to the two-component Bose gas. Here, we extend previous approaches [12, 26] to the most general case where the coupling terms can be position-dependent. Inserting the modal expansion

$$\mathcal{B}(\mathbf{r}) = \sum_\nu \left(\begin{bmatrix} u_{1\nu}(\mathbf{r}) \\ v_{1\nu}(\mathbf{r}) \\ u_{2\nu}(\mathbf{r}) \\ v_{2\nu}(\mathbf{r}) \end{bmatrix} \hat{b}_\nu + \begin{bmatrix} v_{1\nu}^*(\mathbf{r}) \\ u_{1\nu}^*(\mathbf{r}) \\ v_{2\nu}^*(\mathbf{r}) \\ u_{2\nu}^*(\mathbf{r}) \end{bmatrix} \hat{b}_\nu^\dagger \right), \tag{27}$$

with $\hat{b}_\nu$ the annihilation operator of an elementary excitation of the coupled two-component Bose gas, into Eq. (23), we find

$$\hat{H}^{(2)} = \frac{1}{2}\sum_\nu E_\nu \left(\hat{b}_\nu^\dagger \hat{b}_\nu + \hat{b}_\nu \hat{b}_\nu^\dagger\right), \tag{28}$$

provided that the wave functions fulfill the so-called coupled Bogoliubov equations:

$$\begin{bmatrix} \mathcal{L}_1^{\rm GP} & \Gamma \\ \Gamma^* & \mathcal{L}_2^{\rm GP} \end{bmatrix} \begin{bmatrix} u_{1\nu} \\ v_{1\nu} \\ u_{2\nu} \\ v_{2\nu} \end{bmatrix} = E_\nu \begin{bmatrix} u_{1\nu} \\ v_{1\nu} \\ u_{2\nu} \\ v_{2\nu} \end{bmatrix} \tag{29}$$

and the bi-orthogonality conditions

$$\sum_\sigma \int \mathrm{d}\mathbf{r} \left[u_{\sigma\nu}(\mathbf{r}) u^*_{\sigma\nu'}(\mathbf{r}) - v_{\sigma\nu}(\mathbf{r}) v^*_{\sigma\nu'}(\mathbf{r}) \right] = \delta_{\nu\nu'} \tag{30}$$

$$\sum_\sigma \int \mathrm{d}\mathbf{r} \left[u_{\sigma\nu}(\mathbf{r}) v_{\sigma\nu'}(\mathbf{r}) - v_{\sigma\nu}(\mathbf{r}) u_{\sigma\nu'}(\mathbf{r}) \right] = 0\,. \tag{31}$$

These modes (indexed by ν), being of bosonic nature, satisfy the Bose commutation rules $[\hat{b}_{\sigma\nu}, \hat{b}^\dagger_{\sigma'\nu'}] = \delta_{\sigma\sigma'}\delta_{\nu\nu'}$ and $[\hat{b}_{\sigma\nu}, \hat{b}_{\sigma'\nu'}] = 0$.

3. *Orthogonal field operator*

The extended Bogoliubov approach finally requires the orthogonalization of the field operator $\hat{B}_\sigma$ with respect to the (quasi-)condensate wave function $\psi_\sigma(\mathbf{r}) \equiv \mathrm{e}^{i\theta_\sigma}\sqrt{n_\sigma}$ (see Refs. [48, 58]). It amounts to apply the substitution $\hat{B}_\sigma(\mathbf{r}) \to \hat{\Lambda}_\sigma(\mathbf{r})$ with

$$\hat{\Lambda}_\sigma(\mathbf{r}) \equiv \hat{B}_\sigma(\mathbf{r}) - \frac{\psi_\sigma(\mathbf{r})}{N_\sigma} \int \mathrm{d}\mathbf{r}'\ \hat{B}_\sigma(\mathbf{r}')\psi^*_\sigma(\mathbf{r}')\,. \tag{32}$$

We then have

$$\hat{\Lambda}_\sigma(\mathbf{r}) = \sum_\nu \left[u^\perp_{\sigma\nu}(\mathbf{r})\hat{b}_\nu + v^{\perp *}_{\sigma\nu}(\mathbf{r})\hat{b}^\dagger_\nu \right] \tag{33}$$

with

$$u^\perp_{\sigma\nu} \equiv u_{\sigma\nu} - \frac{\psi_\sigma(\mathbf{r})}{N_\sigma} \int \mathrm{d}\mathbf{r}'\ u_{\sigma\nu}(\mathbf{r}')\psi^*_\sigma(\mathbf{r}') \tag{34}$$

$$v^\perp_{\sigma\nu} \equiv v_{\sigma\nu} - \frac{\psi^*_\sigma(\mathbf{r})}{N_\sigma} \int \mathrm{d}\mathbf{r}'\ v_{\sigma\nu}(\mathbf{r}')\psi_\sigma(\mathbf{r}')\,. \tag{35}$$

According to Eqs. (18) and (19), the orthogonal field operators $\hat{\Lambda}_\sigma$ satisfy the modified commutation rules

$$[\hat{\Lambda}_\sigma(\mathbf{r}), \hat{\Lambda}_{\sigma'}(\mathbf{r}')] = 0 \tag{36}$$

$$[\hat{\Lambda}_\sigma(\mathbf{r}), \hat{\Lambda}^\dagger_{\sigma'}(\mathbf{r}')] = \delta_{\sigma\sigma'} \left[\delta(\mathbf{r}-\mathbf{r}') - \frac{\psi_\sigma(\mathbf{r})\psi^*_\sigma(\mathbf{r}')}{N_\sigma} \right]. \tag{37}$$

The solutions of the non-Hermitian eigenvalue problem (29), together with the bi-orthogonality conditions (30)-(31) and the orthogonalization process (34)-(35), determine the excitation spectrum of the two-component Bose gas in the weakly-interacting regime. A mode ν describes a coupled two-component elementary excitation (Bogoliubov quasiparticle) of the mixture. The energy and wave functions of these excitations are E_ν and $\{u^\perp_{1\nu}(\mathbf{r}), v^\perp_{1\nu}(\mathbf{r}), u^\perp_{2\nu}(\mathbf{r}), v^\perp_{2\nu}(\mathbf{r})\}$, respectively. They can be determined numerically, or analytically in certain cases. All physical observables can then be constructed by expansion on the corresponding basis.

D. Correlation functions

We now consider the correlation properties of observable quantities, namely the phases and the densities of the two-component Bose gas. These quantities can be measured independently for each component in experiments with ultracold atoms, using a gaseous mixture of a single bosonic atom prepared in two different internal states [7–9, 55] and internal-state dependent imaging techniques [59]. The density profiles, fluctuations and correlation functions of each component are then found directly from the images [60, 61]. The phase fluctuations and correlation functions of each component are found by time-of-flight [62, 63] or Bragg spectroscopy [64–66] techniques. The total and relative density profiles are then obtained by addition or subtraction of those of each component, which also provides their fluctuations and correlation functions. Finally, the correlation function of the relative phase, $\theta = \theta_1 - \theta_2$ can be found using matter-wave interference techniques [9, 30].

For each component σ, the phase correlation function is

$$G^\sigma_\theta(\mathbf{r},\mathbf{r}') \equiv \langle \hat{\theta}_\sigma(\mathbf{r})\hat{\theta}_\sigma(\mathbf{r}')\rangle - \langle \hat{\theta}_\sigma(\mathbf{r})\rangle\langle \hat{\theta}_\sigma(\mathbf{r}')\rangle = -\frac{\langle :(\hat{\Lambda}_\sigma - \hat{\Lambda}^\dagger_\sigma)(\hat{\Lambda}'_\sigma - \hat{\Lambda}^{\dagger\prime}_\sigma):\rangle}{4\sqrt{n_\sigma\, n'_\sigma}}\,, \tag{38}$$

where the nude (resp. primed) quantities are evaluated at point $\mathbf{r}$ (resp. $\mathbf{r}'$). The operator $:\ \ :$ represents normal ordering with respect to the orthogonal field operators $\hat{\Lambda}$ and $\hat{\Lambda}^\dagger$, which is used to avoid unphysical divergences [48]. Similarly, the density correlation function is

$$G^\sigma_n(\mathbf{r},\mathbf{r}') \equiv \langle n_\sigma(\mathbf{r}) n_\sigma(\mathbf{r}')\rangle - \langle n_\sigma(\mathbf{r})\rangle\langle n_\sigma(\mathbf{r}')\rangle = \sqrt{n_\sigma\, n'_\sigma}\,\langle :(\hat{\Lambda}_\sigma + \hat{\Lambda}^\dagger_\sigma)(\hat{\Lambda}'_\sigma + \hat{\Lambda}^{\dagger\prime}_\sigma):\rangle\,. \tag{39}$$

Using the expansion of the orthogonal field operator into the basis of orthogonal Bogoliubov modes, Eq. (33), and the usual auxiliary wave functions [84]

$$f^{\rm p}_{\sigma\nu}(\mathbf{r}) = u^\perp_{\sigma\nu}(\mathbf{r}) - v^\perp_{\sigma\nu}(\mathbf{r})\,, \tag{40}$$

$$f^{\rm m}_{\sigma\nu}(\mathbf{r}) = u^\perp_{\sigma\nu}(\mathbf{r}) + v^\perp_{\sigma\nu}(\mathbf{r})\,, \tag{41}$$

we then get the following explicit expressions after some algebraic calculations:

$$G^\sigma_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{2\sqrt{n_\sigma n'_\sigma}} \sum_\nu \mathcal{R}e\left[f^{\rm p}_{\sigma\nu} f^{\rm p\prime *}_{\sigma\nu} N_\nu - f^{\rm p}_{\sigma\nu} v^{\perp\prime *}_{\sigma\nu} \right] \tag{42}$$

and

$$G^\sigma_n(\mathbf{r},\mathbf{r}') = 2\sqrt{n_\sigma n'_\sigma} \sum_\nu \mathcal{R}e\left[f^{\rm m}_{\sigma\nu} f^{\rm m\prime *}_{\sigma\nu} N_\nu + f^{\rm m}_{\sigma\nu} v^{\perp\prime *}_{\sigma\nu} \right], \tag{43}$$

where

$$N_\nu = \frac{1}{\exp(E_\nu / k_{\rm B}T) - 1} \tag{44}$$

is the thermal population of mode ν, according to the Bose-Einstein statistical distribution. Note that expressions (42) and (43) are symmetric in $(\mathbf{r},\mathbf{r}')$. This can be checked by noting that the commutation rule $[\hat{\Lambda}_\sigma(\mathbf{r}),\hat{\Lambda}_\sigma(\mathbf{r}')]=0$ [see Eq. (36)] implies the relation $\sum_\nu u^\perp_{\sigma\nu}(\mathbf{r})v^{\perp *}_{\sigma\nu}(\mathbf{r}')=\sum_\nu u^\perp_{\sigma\nu}(\mathbf{r}')v^{\perp *}_{\sigma\nu}(\mathbf{r})$.

The two-point correlation function of the relative phase is defined by the same formula as Eq. (38) with θ_σ replaced by $\theta=\theta_1-\theta_2$. The same calculation strategy yields

$$G_\theta(\mathbf{r},\mathbf{r}')=\frac{1}{2}\sum_\nu \mathcal{R}e\Big[\Big(\frac{f^{\mathrm{p}}_{1\nu}}{\sqrt{n_1}}-\frac{f^{\mathrm{p}}_{2\nu}}{\sqrt{n_2}}\Big)\Big(\frac{f^{\mathrm{p}\prime}_{1\nu}}{\sqrt{n_1'}}-\frac{f^{\mathrm{p}\prime}_{2\nu}}{\sqrt{n_2'}}\Big)^* N_\nu \\ -\Big(\frac{f^{\mathrm{p}}_{1\nu}}{\sqrt{n_1}}-\frac{f^{\mathrm{p}}_{2\nu}}{\sqrt{n_2}}\Big)\Big(\frac{v^{\perp\prime}_{1\nu}}{\sqrt{n_1'}}-\frac{v^{\perp\prime}_{2\nu}}{\sqrt{n_2'}}\Big)^*\Big]. \quad (45)$$

Having developed a general formalism for calculating the excitation modes of the two-component Bose gas with arbitrary one- and two-body couplings, and established general formulas for the density and phase correlation functions, we explicitly calculate these quantities for various homogeneous cases in the next section.

III. HOMOGENEOUS SYSTEMS

In this section, we consider a homogeneous system, where all potentials (V_1 and V_2) and coupling terms (g_1, g_2, g_{12} and Ω) in Hamiltonians (2) and (3) are independent of the position. Assuming that the potentials V_1 and V_2 are equal [85], it can be assumed without loss of generality that $V_1=V_2=0$. This case allows for analytical calculations and contains the main physical effects discussed below. Hereafter, we first rewrite the formalism of Sec. II in a form adapted to the homogeneous case (Sec. III A) and then focus on three cases corresponding to different couplings (Sec. III B, III C, and III D).

A. Meanfield equations

Since all derivative terms in the Euler-Lagrange equations (9) and (10) vanish in the homogeneous case, it immediately follows from Eq. (10) that $\theta-\alpha=0$ or π if $\Omega=\Omega_0 e^{-i\alpha}\neq 0$. Inserting these two solutions into the meanfield version of Eq. (6), we find that $\theta=\alpha$ is a maximum of E_{MF} and is thus an unstable solution. The stable solution is $\theta=\alpha+\pi$, which is a minimum of E_{MF}. For instance, the two components are in phase (resp. out of phase) when $\Omega\in\mathbb{R}^-$ (resp. $\Omega\in\mathbb{R}^+$). If $\Omega=0$, the relative phase θ is not a determined quantity as already discussed in the first paragraph of Sec. II A. Inserting the stable solution into Eq. (9), we then find

$$g_1 n_1+g_{12}n_2-\mu-\frac{\hbar\Omega_0}{2}\sqrt{\frac{n_2}{n_1}}=0 \quad (46)$$

$$g_2 n_2+g_{12}n_1-\mu-\frac{\hbar\Omega_0}{2}\sqrt{\frac{n_1}{n_2}}=0 \quad (47)$$

and $n_1+n_2=n=N/\mathcal{V}$ with N the total number of particles and $\mathcal{V}$ the volume of the system. In all cases discussed below, we assume that the parameters are such that the two components are miscible, i.e. there exists a solution of Eqs. (46) and (47) with $n_1>0$ and $n_2>0$. The corresponding conditions are discussed below for some particular cases.

Translation invariance ensures that the Bogoliubov modes are the plane waves

$$u_{\sigma\mathbf{k}}(\mathbf{r})=\frac{1}{\sqrt{\mathcal{V}}}\tilde{u}_{\sigma\mathbf{k}}e^{i\mathbf{k}.\mathbf{r}} \quad (48)$$

$$v_{\sigma\mathbf{k}}(\mathbf{r})=\frac{1}{\sqrt{\mathcal{V}}}\tilde{v}_{\sigma\mathbf{k}}e^{i\mathbf{k}.\mathbf{r}}, \quad (49)$$

$$f^{p,m}_{\sigma\mathbf{k}}(\mathbf{r})=\frac{1}{\sqrt{\mathcal{V}}}\tilde{f}^{p,m}_{\sigma\mathbf{k}}e^{i\mathbf{k}.\mathbf{r}}, \quad (50)$$

where we label the modes by the wave vector $\mathbf{k}$ (instead of ν). In the following, we omit the tilde sign to simplify the notations. Then, the amplitudes $u_{1\mathbf{k}}$, $v_{1\mathbf{k}}$, $u_{2\mathbf{k}}$, and $v_{2\mathbf{k}}$ are the solutions of the eigenproblem (29) for the diagonal blocks

$$\mathcal{L}^{GP}_\sigma=\begin{bmatrix}+\mathbf{A}_{\sigma\mathbf{k}} & +g_\sigma n_\sigma\\ -g_\sigma n_\sigma & -\mathbf{A}_{\sigma\mathbf{k}}\end{bmatrix}, \quad (51)$$

with $\mathbf{A}_{\sigma\mathbf{k}}=\epsilon_{\mathbf{k}}+2g_\sigma n_\sigma+g_{12}n_{\bar\sigma}-\mu$ where $\epsilon_{\mathbf{k}}=\hbar^2\mathbf{k}^2/2m$ is the free-particle dispersion relation, and for the off-diagonal blocks

$$\Gamma=\begin{bmatrix}+g_{12}\sqrt{n_1n_2}-\hbar\Omega_0/2 & +g_{12}\sqrt{n_1n_2}\\ -g_{12}\sqrt{n_1n_2} & -g_{12}\sqrt{n_1n_2}+\hbar\Omega_0/2\end{bmatrix}. \quad (52)$$

The biorthogonality conditions (30) and (31) reduce to

$$\sum_{\sigma=1,2}\left(|u_{\sigma\mathbf{k}}|^2-|v_{\sigma\mathbf{k}}|^2\right)=1. \quad (53)$$

Note that since the classical fields ϕ_σ is homogeneous and the Bogoliubov wave function $u_{\sigma\mathbf{k}}$ and $v_{\sigma\mathbf{k}}$ are plane waves the orthogonalization procedure of Eqs. (30) and (31) is irrelevant for $\mathbf{k}\neq 0$.

Finally, the correlation functions introduced in Sec. II D are found by inserting Eqs. (48) and (49) into Eq. (42) and (43), which yields the following explicit formulas: For the phase correlation function of component σ,

$$G^\sigma_\theta(\mathbf{r},\mathbf{r}')=\frac{1}{2n_\sigma\mathcal{V}}\sum_{\mathbf{k}\neq 0}\Big[|f^p_{\sigma\mathbf{k}}|^2N_{\mathbf{k}}-f^p_{\sigma\mathbf{k}}v^*_{\sigma\mathbf{k}}\Big]\cos\left[\mathbf{k}.(\mathbf{r}-\mathbf{r}')\right]; \quad (54)$$

For the density correlation function of component σ,

$$G^\sigma_n(\mathbf{r},\mathbf{r}')=\frac{2n_\sigma}{\mathcal{V}}\sum_{\mathbf{k}\neq 0}\Big[|f^m_{\sigma\mathbf{k}}|^2N_{\mathbf{k}}+f^m_{\sigma\mathbf{k}}v^*_{\sigma\mathbf{k}}\Big]\cos\left[\mathbf{k}.(\mathbf{r}-\mathbf{r}')\right]. \quad (55)$$

Similarly, the correlation function of the relative phase is

$$G_\theta(\mathbf{r},\mathbf{r}')=\frac{1}{2\mathcal{V}}\sum_{\mathbf{k}\neq 0}\Big[\Big|\frac{f^p_{1\mathbf{k}}}{\sqrt{n_1}}-\frac{f^p_{2\mathbf{k}}}{\sqrt{n_2}}\Big|^2 N_{\mathbf{k}} \\ -\Big(\frac{f^p_{1\mathbf{k}}}{\sqrt{n_1}}-\frac{f^p_{2\mathbf{k}}}{\sqrt{n_2}}\Big)\Big(\frac{v_{1\mathbf{k}}}{\sqrt{n_1}}-\frac{v_{2\mathbf{k}}}{\sqrt{n_2}}\Big)^*\Big]\cos\left[\mathbf{k}.(\mathbf{r}-\mathbf{r}')\right]. \tag{56}$$

Note that, for simplicity, we have indicated only $\mathbf{k}\neq 0$ below the sum symbols of Eqs. (54), (55) and (56). As a matter of fact, we will see that in general the Bogoliubov spectrum displays two branches, over which the sums should be performed.

B. Case of two-body coupling ($g_{12}\neq 0$ and $\Omega = 0$)

Here we consider the case where the two Bose components are coupled via two-body interactions ($g_{12}\neq 0$) with no one-body interaction ($\Omega = 0$). This case describes for instance a mixture of gaseous Bose-Einstein (quasi-)condensates with atoms in two different internal states and confined in the same trap, as realized with ^{87}Rb atoms in Refs. [7–9, 55, 69, 70]. The miscibility condition of the two components requires that the intra-component couplings exceed the inter-component couplings [14], i.e. $|g_{12}| < g_1,\ g_2$. Hereafter, we provide useful details about the calculations, and are more brief in the next sections where the same techniques is used.

1. *Meanfield background and Bogoliubov excitations*

The background densities and the chemical potential are found straightforwardly from Eqs. (46) and (47), which yields

$$n_\sigma = \frac{N}{\mathcal{V}}\frac{g_{\bar\sigma}-g_{12}}{g_1+g_2-2g_{12}} \tag{57}$$

$$\mu = \frac{N}{\mathcal{V}}\frac{g_1g_2-g_{12}^2}{g_1+g_2-2g_{12}}\,. \tag{58}$$

Then, the Bogoliubov excitations are the solutions of

$$\begin{bmatrix} \epsilon_k+g_1n_1 & g_1n_1 & g_{12}\sqrt{n_1n_2} & g_{12}\sqrt{n_1n_2} \\ -g_1n_1 & -\epsilon_k-g_1n_1 & -g_{12}\sqrt{n_1n_2} & -g_{12}\sqrt{n_1n_2} \\ g_{12}\sqrt{n_1n_2} & g_{12}\sqrt{n_1n_2} & \epsilon_k+g_2n_2 & g_2n_2 \\ -g_{12}\sqrt{n_1n_2} & -g_{12}\sqrt{n_1n_2} & -g_2n_2 & -\epsilon_k-g_2n_2 \end{bmatrix}\begin{bmatrix} u_{1k}\\ v_{1k}\\ u_{2k}\\ v_{2k}\end{bmatrix} = E_k\begin{bmatrix} u_{1k}\\ v_{1k}\\ u_{2k}\\ v_{2k}\end{bmatrix}, \tag{59}$$

which corresponds to Eq. (29) where we have inserted Eqs. (51) and (52) for the considered case. Note that using Eqs. (46) and (47), we find $\mathbf{A}_{\sigma\mathbf{k}} = \epsilon_{\mathbf{k}} + g_\sigma n_\sigma$. By taking the sum and difference of the first two rows on the one hand, and of the last two rows on the other hand, we find

$$E_{\mathbf{k}}f^{\mathrm{m}}_{\sigma\mathbf{k}} = \epsilon_{\mathbf{k}}f^{\mathrm{p}}_{\sigma\mathbf{k}} \tag{60}$$

$$E_{\mathbf{k}}f^{\mathrm{p}}_{\sigma\mathbf{k}} = (\epsilon_{\mathbf{k}}+2g_\sigma n_\sigma)f^{\mathrm{m}}_{\sigma\mathbf{k}} + 2g_{12}\sqrt{n_\sigma n_{\bar\sigma}}f^{\mathrm{m}}_{\bar\sigma\mathbf{k}}\,. \tag{61}$$

Inserting the first equation into the second one, and using the biorthogonality condition $f^{\mathrm{p}}_{1\mathbf{k}}f^{\mathrm{m}}_{1\mathbf{k}}+f^{\mathrm{p}}_{2\mathbf{k}}f^{\mathrm{m}}_{2\mathbf{k}}=1$ [which is equivalent to Eq. (53)] to eliminate the $f^{\mathrm{p}}_{\sigma\mathbf{k}}$ functions, we then find

$$[E_{\mathbf{k}}^2-\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+2g_\sigma n_\sigma)]f^{\mathrm{m}}_{\sigma\mathbf{k}} = 2\epsilon_{\mathbf{k}}g_{12}\sqrt{n_1n_2}f^{\mathrm{m}}_{\bar\sigma\mathbf{k}} \tag{62}$$

$$E_{\mathbf{k}}(f^{\mathrm{m}}_{1\mathbf{k}})^2+E_{\mathbf{k}}(f^{\mathrm{m}}_{2\mathbf{k}})^2 = \epsilon_{\mathbf{k}}\,. \tag{63}$$

The Bogoliubov spectrum is found from the ratio of the two avatars of Eq. (62) corresponding to $\sigma = 1$ and $\sigma = 2$ respectively. Using Eqs. (57) and (58), it yields

$$E^\pm_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}\left(\epsilon_{\mathbf{k}}+g_1n_1+g_2n_2\pm\Delta\right)} \tag{64}$$

with $\Delta = |g_{12}|(n_1+n_2)$, where here we retain only the positive energy solutions, which correspond to excitations of the two-component Bose gas. The spectrum is plotted in Fig. 2 for a case with $g_{12}\neq 0$. It is composed of two branches (labelled by $\pm$), which are distinct if and only if $g_{12}\neq 0$. Each branch shows the usual Bogoliubov dispersion relation: For $\epsilon_{\mathbf{k}} \ll g_1n_1+g_2n_2\pm\Delta$, it is phonon-like, $E^\pm_{\mathbf{k}}\simeq c^\pm\hbar k$ with $c^\pm = \sqrt{(g_1n_1+g_2n_2\pm\Delta)/2m}$ the sound velocities; For $\epsilon_{\mathbf{k}} \gg g_1n_1+g_2n_2\pm\Delta$, it is free-particle-like, $E^\pm_{\mathbf{k}}\simeq\epsilon_{\mathbf{k}}+(g_1n_1+g_2n_2\pm\Delta)/2$. In particular, the quantity Δ is the separation of the two branches in the high-energy limit.

In order to determine the Bogoliubov wavefunctions, we replace $E^\pm_{\mathbf{k}}$ in Eq. (62) by its expression (64), which yields (for $g_{12}\neq 0$)

$$\left[-\zeta\pm\sqrt{\zeta^2+1}\right]f^{\mathrm{m}\pm}_{1\mathbf{k}} = \mathrm{sgn}(g_{12})\times f^{\mathrm{m}\pm}_{2\mathbf{k}} \tag{65}$$

$$\left[+\zeta\pm\sqrt{\zeta^2+1}\right]f^{\mathrm{m}\pm}_{2\mathbf{k}} = \mathrm{sgn}(g_{12})\times f^{\mathrm{m}\pm}_{1\mathbf{k}} \tag{66}$$

where $\zeta = (g_1n_1-g_2n_2)/2|g_{12}|\sqrt{n_1n_2}$. Using Eqs. (60) and (63), we then find (for $g_{12}\neq 0$)

$$f^{\mathrm{m}\pm}_{1\mathbf{k}} = \left[\frac{\epsilon_{\mathbf{k}}}{2E^\pm_{\mathbf{k}}\left(1+\zeta^2\mp\zeta\sqrt{\zeta^2+1}\right)}\right]^{1/2} \tag{67}$$

$$f^{\mathrm{p}\pm}_{1\mathbf{k}} = \left[\frac{E^\pm_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}\left(1+\zeta^2\mp\zeta\sqrt{\zeta^2+1}\right)}\right]^{1/2} \tag{68}$$

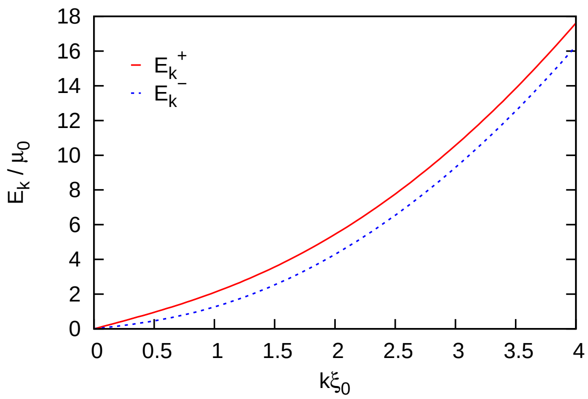

FIG. 2. Bogoliubov spectrum of the coupled excitations in a homogeneous two-component Bose gas with $g_{12} \neq 0$ and $\Omega = 0$. Plotted are the two energy branches $E^{\pm}_{\mathbf{k}}$ [Eq. (64)] in the case $g_1 = g_2$ and $|g_{12}| = 0.7g_1$. Here $\mu_0 = g_1 N/2\mathcal{V}$ is the chemical potential in the absence of any coupling, and $\xi_0 = \hbar/\sqrt{2m\mu_0}$ is the corresponding healing length. In the low-energy limit, the two branches are phonon-like, $E^{\pm}_{\mathbf{k}} \simeq c^{\pm}\hbar k$. In the high-energy limit, they are free-particle like, $E^{\pm}_{\mathbf{k}} \simeq \epsilon_{\mathbf{k}} + (g_1 n_1 + g_2 n_2 \pm \Delta)/2$ and are separated by a constant value Δ.

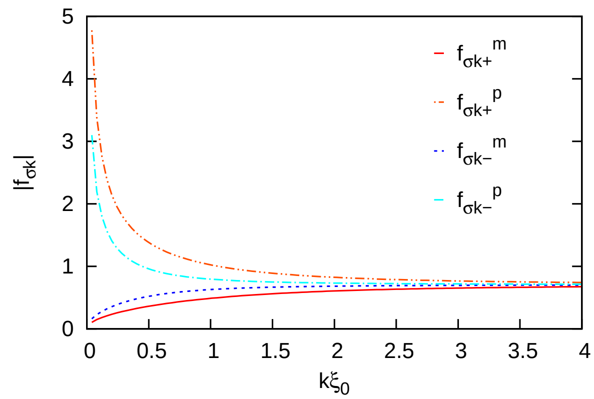

FIG. 3. Amplitudes of the wavefunctions $f^{\mathrm{p,m}}_{\sigma\mathbf{k}}$ of the coupled Bogoliubov excitations for a homogeneous two-component Bose gas with $g_{12} \neq 0$ and $\Omega = 0$. Plotted are the absolute values, $|f^{\mathrm{p,m}}_{\sigma\mathbf{k}}|$ [see Eqs. (67) to (70)] for the same parameters as in Fig. 2. In particular since $g_1 = g_2$, we have $\zeta = 0$ [see below Eqs. (65) and (66)] and the absolute values are independent of the component σ and of the sign of g_{12}. For $g_{12} > 0$, the excitations are off phase in the lower branch ($E^{-}_{\mathbf{k}}$) and in phase for the upper branch ($E^{+}_{\mathbf{k}}$). For $g_{12} < 0$, the excitations are off phase in the upper branch ($E^{+}_{\mathbf{k}}$) and in phase for the lower branch ($E^{-}_{\mathbf{k}}$).

$$f^{\mathrm{m}\pm}_{2\mathbf{k}} = \pm\mathrm{sgn}(g_{12})\times\left[\frac{\epsilon_{\mathbf{k}}}{2E^{\pm}_{\mathbf{k}}\left(1+\zeta^2 \pm \zeta\sqrt{\zeta^2+1}\right)}\right]^{1/2} \quad (69)$$

$$f^{\mathrm{p}\pm}_{2\mathbf{k}} = \pm\mathrm{sgn}(g_{12})\times\left[\frac{E^{\pm}_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}\left(1+\zeta^2 \pm \zeta\sqrt{\zeta^2+1}\right)}\right]^{1/2} \quad (70)$$

where we have set the (arbitrary) sign of f^{m}_1 to be positive. The moduli of these functions are plotted in Fig. 3. In the following, we will omit the branch labels ($\pm$) in the functions $f^{\mathrm{p,m}}_{\sigma\mathbf{k}}$ for simplicity , except when necessary. It follows from these equations that, for a given component σ, the $f^{\mathrm{m}}_{\sigma}(\mathbf{r})$ and $f^{\mathrm{p}}_{\sigma}(\mathbf{r})$ wavefunctions are always in phase [i.e. $f^{\mathrm{m}}_{\sigma\mathbf{k}}f^{\mathrm{p}}_{\sigma\mathbf{k}} > 0$; see also Eq. (60)]. For $g_{12} > 0$, the modes associated to the components 1 ($f^{\mathrm{m}}_{1\mathbf{k}}, f^{\mathrm{p}}_{1\mathbf{k}}$) and 2 ($f^{\mathrm{m}}_{2\mathbf{k}}, f^{\mathrm{p}}_{2\mathbf{k}}$) are off phase ($f^{\mathrm{p}}_{1\mathbf{k}}f^{\mathrm{p}}_{2\mathbf{k}} < 0$ and $f^{\mathrm{m}}_{1\mathbf{k}}f^{\mathrm{m}}_{2\mathbf{k}} < 0$) in the lower ($-$) branch and in phase ($f^{\mathrm{p}}_{1\mathbf{k}}f^{\mathrm{p}}_{2\mathbf{k}} > 0$ and $f^{\mathrm{m}}_{1\mathbf{k}}f^{\mathrm{m}}_{2\mathbf{k}} > 0$) in the upper ($+$) branch (and the other way round for $g_{12} < 0$). It can be traced to the fact that for repulsive inter-component interactions ($g_{12} > 0$), off-phase density fluctuations ($f^{\mathrm{m}}_{1\mathbf{k}}f^{\mathrm{m}}_{2\mathbf{k}} < 0$) cost less interaction energy than in-phase density fluctuations (and the other way round for $g_{12} < 0$).

In the particular case where the two components are decoupled, i.e. for $g_{12} = 0$, the spectrum shows twofold degeneracy (there is also a trivial $+\mathbf{k} \leftrightarrow -\mathbf{k}$ degeneracy, which we disregard here). The two branches of the spectrum are identical and correspond to the usual single-particle Bogoliubov spectrum, $E^{\pm}_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}} + 2\mu)}$. This holds even for $g_1 \neq g_2$ because the meanfield background is identical for the two Bose gases, i.e. $g_1 n_1 = g_2 n_2 = \mu$ [see Eqs. (57) and (58) with $g_{12} = 0$]. In this case, each branch can be ascribed to elementary excitations in one of the components, so that $f^{\mathrm{m}}_{\sigma\mathbf{k}} = 1/f^{\mathrm{p}}_{\sigma\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}/E_{\mathbf{k}}}$ and $f^{\mathrm{m}}_{\bar{\sigma}\mathbf{k}} = f^{\mathrm{p}}_{\bar{\sigma}\mathbf{k}} = 0$.

2. *Fluctuations and correlations*

The phase and density correlations in each component σ are determined by the $f^{\mathrm{p}}_{\sigma\mathbf{k}}$ and $f^{\mathrm{m}}_{\sigma\mathbf{k}}$ functions [see Eqs. (54) and (55)]. Due to the similarity of the dispersion relation and formulas for the $f^{p,m}_{\sigma\mathbf{k}}$ functions with those of a single-component Bose gas, each component behaves as an effective single-component gas. The effective parameters however depend on all coupling parameters g_1, g_2 and g_{12} and are in general different for the two components (if $g_1 \neq g_2$). Then, the density fluctuations remain small for strong-enough interaction parameters and low temperatures in any dimension. In contrast, the behavior of the phase fluctuations strongly depends on the dimension, owing to the $1/\sqrt{|\mathbf{k}|}$ divergence of the $f^{\mathrm{p}}_{\sigma\mathbf{k}}$ functions. In three dimensions, the two components form true Bose-Einstein condensates with intra-component phase coherence. In lower dimensions, they form quasi-condensates with strong intra-component phase fluctuations.

The fluctuations of the relative phase, which we detail more here, follow the same behavior. Indeed, using Eqs. (40), (41), and (60), the correlation function for the relative phase [Eq. (56)] reduces to

$$G_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{4\mathcal{V}} \sum_{\mathbf{k}\neq 0,\pm} \Big| \frac{f^{\mathrm{p}\pm}_{1\mathbf{k}}}{\sqrt{n_1}} - \frac{f^{\mathrm{p}\pm}_{2\mathbf{k}}}{\sqrt{n_2}} \Big|^2 \Big\{ 2N_\mathbf{k} + \Big(1 - \frac{\epsilon_\mathbf{k}}{E^\pm_\mathbf{k}}\Big) \Big\} \times \cos\left[\mathbf{k}.(\mathbf{r}-\mathbf{r}')\right] . \tag{71}$$

where the sum runs over all nonzero values of $\mathbf{k}$ and over the two branches $\pm$. This correlation function has a quantum contribution corresponding to the $|f^{\mathrm{p}}_{1\mathbf{k}}/\sqrt{n_1} - f^{\mathrm{p}}_{2\mathbf{k}}/\sqrt{n_2}|^2(1-\epsilon_\mathbf{k}/E_\mathbf{k})$ terms, and a thermal contribution, corresponding to the $|f^{\mathrm{p}}_{1\mathbf{k}}/\sqrt{n_1} - f^{\mathrm{p}}_{2\mathbf{k}}/\sqrt{n_2}|^2 N_\mathbf{k}$ terms. Since $|f^{\mathrm{p}}_{1\mathbf{k}}/\sqrt{n_1} - f^{\mathrm{p}}_{2\mathbf{k}}/\sqrt{n_2}| \neq 0$ at least in the off-phase branch (corresponding to the lower branch $E^-_\mathbf{k}$ for $g_{12} > 0$ and to the upper branch $E^+_\mathbf{k}$ for $g_{12} < 0$), the low-k term diverges as $1/k$ as for single-component Bose gases [40, 41]. In three dimensions and low temperature, the two components are thus mutually coherent. Conversely, in lower dimensions, they show no true long-range mutual phase coherence.

To further discuss the effect of inter-component two-body interactions, let us restrict ourselves, for the sake of simplicity, to the case where the two intra-component couplings are equal (i.e. $g_1 = g_2 \equiv g$). Then, we have $n_1 = n_2 = n/2$ (with $n = n_1 + n_2$ the total density) and $\zeta = 0$, so that $f^{\mathrm{p}}_{1\mathbf{k}} = f^{\mathrm{p}}_{2\mathbf{k}}$ for the in-phase branch and $f^{\mathrm{p}}_{1\mathbf{k}} = -f^{\mathrm{p}}_{2\mathbf{k}}$ for the off-phase branch [see Eqs. (68) and (70)]. It results that, as can be expected, only the off-phase branch contributes to the correlation function of the relative phase. This contribution depends quantitatively on whether the off-phase branch corresponds to the upper or the lower branch, i.e. on the sign of g_{12}. In the contributing off-phase branch, we have $|f^{\mathrm{p}}_{1\mathbf{k}}/\sqrt{n_1} - f^{\mathrm{p}}_{2\mathbf{k}}/\sqrt{n_2}|^2 = 2E^{\mathrm{off}}_\mathbf{k}/n\epsilon_\mathbf{k}$ where $E^{\mathrm{off}}_\mathbf{k} = \sqrt{\epsilon_\mathbf{k}[\epsilon_\mathbf{k} + (g-g_{12})n]}$ is the energy of the mode ($E^-_\mathbf{k}$ for $g_{12} > 0$ and $E^+_\mathbf{k}$ for $g_{12} < 0$). Hence, the quantum contribution to the relative phase fluctuations is weaker for repulsive inter-component interactions ($g_{12} > 0$) than for attractive inter-component interactions ($g_{12} < 0$). The behavior of the thermal contribution is more involved because, although the amplitude $|f^{\mathrm{p}}_{1\mathbf{k}}/\sqrt{n_1} - f^{\mathrm{p}}_{2\mathbf{k}}/\sqrt{n_2}|^2$ is smaller for $g_{12} > 0$, the populations $N_\mathbf{k}$ of the off-phase branch excitations are larger since it is the lower branch. To determine the overall behavior of the relative phase fluctuations, it is worth rewriting Eq. (71) into the more compact form

$$G_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{2n\mathcal{V}} \sum_{\mathbf{k}\neq 0} \left[\frac{E^{\mathrm{off}}_\mathbf{k}}{\epsilon_\mathbf{k}} \coth\left(\frac{E^{\mathrm{off}}_\mathbf{k}}{2k_{\mathrm{B}}T}\right) - 1 \right] \cos[\mathbf{k}.(\mathbf{r}-\mathbf{r}')] \tag{72}$$

where the sum runs over a single-branch corresponding to all values of $\mathbf{k} \neq 0$. Then, since $u\coth(u)$ is an increasing function of u (for $u > 0$) and $E^{\mathrm{off}}_\mathbf{k}$ is a decreasing function of g_{12}, we conclude that the relative phase fluctuations are weaker for $g_{12} > 0$ than for $g_{12} < 0$. In particular, the relative phase fluctuations are maximally suppressed when $g_{12} > 0$ approaches g from below. In other words, in a homogeneous two-component Bose gas, repulsive inter-component interactions reduce relative phase fluctuations while attractive inter-component interactions enhance relative phase fluctuations.

C. Case of one-body coupling ($\Omega \neq 0$ and $g_{12} = 0$)

Let us turn to the opposite case where the two Bose components are coupled via one-body coupling ($\Omega \neq 0$) but not by two-body coupling ($g_{12} = 0$). This case can be realized for instance using two Bose-Einstein condensates in two elongated traps, coupled by non-vanishing quantum tunneling, as considered in Refs. [26, 30].

1. *Meanfield background and Bogoliubov excitations*

In this case, there is no simple general solution to Eqs. (46) and (47). However, straightforward analysis of these equations for $g_{12} = 0$ shows that there is a unique solution, such that $n_\sigma \geq \mu/g_\sigma$ for any $\mu > 0$. For the sake of simplicity, let us assume $g_1 = g_2 \equiv g$. Then, Eqs. (46) and (47) are considerably simplified, and we easily find

$$n_1 = n_2 = \frac{N}{2\mathcal{V}} \tag{73}$$

$$\mu = \frac{gN}{2\mathcal{V}} - \frac{\hbar\Omega_0}{2} . \tag{74}$$

Then, the Bogoliubov excitations are obtained by the same procedure as in Sec. III B. We find a two-branch spectrum given by the equations

$$E^-_\mathbf{k} = \sqrt{\epsilon_\mathbf{k}\left(\epsilon_\mathbf{k} + gn\right)} \tag{75}$$

$$E^+_\mathbf{k} = \sqrt{\left(\epsilon_\mathbf{k} + \hbar\Omega_0\right)\left(\epsilon_\mathbf{k} + \hbar\Omega_0 + gn\right)} , \tag{76}$$

where $n = n_1 + n_2$ is the total density. The spectrum is plotted in Fig. 4 for a case with $\Omega_0 \neq 0$. The two branches (labelled by $\pm$) are distinct if and only if $\Omega_0 \neq 0$. The lower branch ($-$) shows the usual (ungapped) Bogoliubov dispersion relation: For $\epsilon_\mathbf{k} \ll gn$, it is phonon-like, $E^-_\mathbf{k} \simeq c\hbar k$ with $c = \sqrt{gn/2m}$ the sound velocity; For $\epsilon_\mathbf{k} \gg gn$, it is free-particle-like, $E^-_\mathbf{k} \simeq \epsilon_\mathbf{k} + gn/2$. In contrast, the upper branch ($+$) is gapped and free-particle-like in both low- and high-energy limits: For $\epsilon_\mathbf{k} \ll gn, \hbar\Omega_0$, we have $E^+_\mathbf{k} \simeq \sqrt{\hbar\Omega_0(\hbar\Omega_0 + gn)} + \frac{2\hbar\Omega_0 + gn}{2\sqrt{\hbar\Omega_0(\hbar\Omega_0+gn)}}\epsilon_\mathbf{k}$; For $\epsilon_\mathbf{k} \gg gn, \hbar\Omega_0$, we have $E^+_\mathbf{k} \simeq \epsilon_\mathbf{k} + \hbar\Omega_0 + gn/2$. In particular, the quantity $\sqrt{\hbar\Omega_0(\hbar\Omega_0 + gn)}$ is the gap of the upper branch, while the quantity $\hbar\Omega_0$ is the separation between the two branches in the high-energy limit.

The Bogoliubov wavefunctions are then found following the same procedure as in Sec. III B.

$$f^{\mathrm{m}\pm}_{1\mathbf{k}} = \left[\frac{\epsilon_\mathbf{k} + \hbar\Omega_0/2 \pm \hbar\Omega_0/2}{2E^\pm_\mathbf{k}} \right]^{1/2} \tag{77}$$

$$f^{\mathrm{p}\pm}_{1\mathbf{k}} = \left[\frac{E^\pm_\mathbf{k}}{2\epsilon_\mathbf{k} + \hbar\Omega_0 \pm \hbar\Omega_0} \right]^{1/2} \tag{78}$$

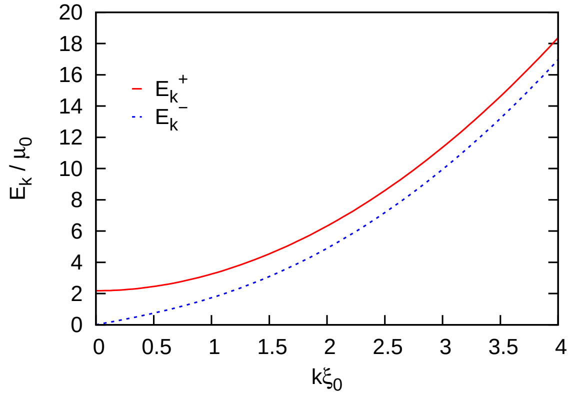

FIG. 4. Bogoliubov spectrum of the coupled excitations in a homogeneous two-component Bose gas with $g_{12}=0$ and $\Omega \neq 0$. Plotted are the two energy branches $E_{\mathbf{k}}^{\pm}$ [Eq. (75)and (76)] in the case $g_1 = g_2$ and $\hbar\Omega_0 = 0.7gn$. Here, $\mu_0 = g_1 N/2\mathcal{V}$ is the chemical potential in the absence of any coupling, and $\xi_0 = \hbar/\sqrt{2m\mu_0}$ is the corresponding healing length. In contrast, for $\Omega_0 \neq 0$, the upper branch ($E_{\mathbf{k}}^{+}$) is gapped and quadratic in both low and high-energy limits.

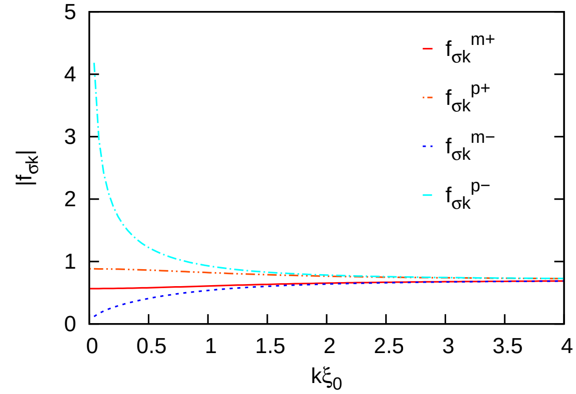

FIG. 5. Amplitudes of the wavefunctions $f_{\sigma\mathbf{k}}^{\mathrm{p,m}}$ of the coupled Bogoliubov excitations for a homogeneous two-component Bose gas with $g_{12}=0$ and $\Omega \neq 0$. Plotted are the absolute values, $|f_{\sigma\mathbf{k}}^{\mathrm{p,m}}|$ [see Eqs. (67) to (70)] for the same parameters as in Fig. 4. Since $g_1 = g_2$, the absolute values are independent of the component σ. The excitations are in phase in the lower branch ($E_{\mathbf{k}}^{-}$) and off phase for the upper branch ($E_{\mathbf{k}}^{+}$).

$$f_{2\mathbf{k}}^{\mathrm{m}\pm} = \mp \left[\frac{\epsilon_{\mathbf{k}} + \hbar\Omega_0/2 \pm \hbar\Omega_0/2}{2E_{\mathbf{k}}^{\pm}}\right]^{1/2} \quad (79)$$

$$f_{2\mathbf{k}}^{\mathrm{p}\pm} = \mp \left[\frac{E_{\mathbf{k}}^{\pm}}{2\epsilon_{\mathbf{k}} + \hbar\Omega_0 \pm \hbar\Omega_0}\right]^{1/2} \quad (80)$$

Their moduli are plotted in Fig. 5. They do not depend on the component σ since we considered only the case where $g_1 = g_2$. As in Sec. III B, for a given component σ, the $f_\sigma^{\mathrm{m}}(\mathbf{r})$ and $f_\sigma^{\mathrm{p}}(\mathbf{r})$ wavefunctions are always in phase. However, the modes associated to the components 1 ($f_{1\mathbf{k}}^{\mathrm{m}}, f_{1\mathbf{k}}^{\mathrm{p}}$) and 2 ($f_{2\mathbf{k}}^{\mathrm{m}}, f_{2\mathbf{k}}^{\mathrm{p}}$) are now off phase in the upper (+) branch and in phase in the lower (−) branch. In the lower (−) branch, each component behaves as a single-component Bose gas with renormalized effective parameters, since the previous Bogoliubov spectrum and wavefunctions are similar to those of a single-component gas. In contrast, the gapped dispersion relation of the upper (+) branch yields a different behavior for the $f_{\sigma\mathbf{k}}^{\mathrm{p}+}$ and $f_{\sigma\mathbf{k}}^{\mathrm{m}+}$ functions. They do not depend much on $\mathbf{k}$ as soon as $\hbar\Omega_0$ and gn are of the same order, and in particular the $f_{\sigma\mathbf{k}}^{\mathrm{p}+}$ functions no longer diverge at low energy, since the gap acts as a low-energy cut-off.

2. *Fluctuations and correlations*

The phase and density fluctuations within one component σ are governed by the $f_{\sigma\mathbf{k}}^{\mathrm{p}}$ and $f_{\sigma\mathbf{k}}^{\mathrm{m}}$ functions. Let us discuss the relative phase fluctuations. Proceeding as in Sec. III B, we can rewrite the correlation function for the relative phase

$$G_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{2\mathcal{V}n}\sum_{\mathbf{k}\neq 0,\pm}\left\{2N_{\mathbf{k}} + \left(1 - \frac{\epsilon_{\mathbf{k}} + \hbar\Omega_0/2 \pm \hbar\Omega_0/2}{E_{\mathbf{k}}^{\pm}}\right)\right\} \times \left|f_{1\mathbf{k}}^{\mathrm{p}\pm} - f_{2\mathbf{k}}^{\mathrm{p}\pm}\right|^2 \cos\left[\mathbf{k}.(\mathbf{r}-\mathbf{r}')\right], \quad (81)$$

making appear the thermal and quantum contributions. As in Sec. III B, only the off-phase branch contributes to $G_\theta(\mathbf{r},\mathbf{r}')$. Here, the off-phase branch is always the upper one, independently of the sign of $\Omega(\mathbf{r})$, or more generally independently of its phase α. Due to the gap of the upper branch, its contribution remains finite and does not suppress mutual phase coherence of the two Bose gases, in any dimension. The one-body coupling thus tends to favor fluctuations of the phases of the components that are in phase. It contrasts with the mean-field phases θ_1 and θ_2, the difference of which is imposed by the sign of $\Omega(\mathbf{r})$, or more generally its phase α [see Sec. III A]. This behavior can by understood from the fact that the one-body coupling tends to impose the difference between the total phases of the two components. Since it is realized at the meanfield level, the phase fluctuations tend to be in phase. By rewriting Eq.(81) into the form

$$G_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{n\mathcal{V}}\sum_{\mathbf{k}\neq 0}\left[\sqrt{\frac{\epsilon_{\mathbf{k}} + gn + \hbar\Omega_0}{\epsilon_{\mathbf{k}} + \hbar\Omega_0}}\coth\left(\frac{E_{\mathbf{k}}^{+}}{2k_{\mathrm{B}}T}\right) - 1\right] \times \cos[\mathbf{k}.(\mathbf{r}-\mathbf{r}')] \quad (82)$$

one can indeed check that since $E_{\mathbf{k}}^{+}$ increases with Ω_0, both $\coth(E_{\mathbf{k}}^{+}/2k_{\mathrm{B}}T)$ and $\sqrt{(\epsilon_{\mathbf{k}} + gn + \hbar\Omega_0)/(\epsilon_{\mathbf{k}} + \hbar\Omega_0)}$ decrease when Ω_0 increases, so that the relative phase fluctuations $G_\theta(\mathbf{r},\mathbf{r})$ decrease when the intensity of the one-body coupling increases. In particular, for a temperature smaller than the gap, $k_{\mathrm{B}}T < \sqrt{\hbar\Omega_0(\hbar\Omega_0 + gn)}$, the quantum fluctuations of the relative phase dominate, and are strongly suppressed for $\hbar\Omega_0 > gn$. These results generalize those of Ref. [26] to the case of a one-body coupling of arbitrary phase.

D. Case of one-body ($\Omega \neq 0$) and two-body ($g_{12} < g_1 = g_2$) couplings

As discussed in the last two sections, the one-body coupling tends to establish the mutual coherence between the two Bose gases and to suppress relative phase fluctuations, while the two-body coupling can reduce or enhance the relative phase fluctuations depending on the sign of g_{12}. Let us now consider the general case where both one-body and two-body couplings are present, and study how they interplay.

1. Meanfield background and Bogoliubov excitations

For the sake of simplicity, we will assume as in Sec. III C that $g_1 = g_2 \equiv g$, which captures the main physics of the problem. Detailed calculations for the general case are shown in the appendix. In the case we consider here, the densities of the two components are equal, $n_1 = n_2$, and Eqs.(46) and (47) yield the chemical potential $\mu = (g + g_{12})n/2 - \hbar\Omega_0/2$, with $n = n_1 + n_2$ the total density. The same procedure as in Sec. III B yields the two-branch spectrum

$$E^-_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}\left(\epsilon_{\mathbf{k}} + gn + g_{12}n\right)} \tag{83}$$

$$E^+_{\mathbf{k}} = \sqrt{\left(\epsilon_{\mathbf{k}} + \hbar\Omega_0\right)\left(\epsilon_{\mathbf{k}} + \hbar\Omega_0 + (g - g_{12})n\right)}\,, \tag{84}$$

which is plotted in Fig. 6. The $(-)$ branch is one of the two branches of the case with only two-body coupling (Sec. III B). In particular, we recover the results of Sec. III B of two separated Bogoliubov-like branches when $\Omega_0 \to 0$. However, if $g_{12} > 0$, the $(-)$ branch yields the previous upper $(+)$ branch, and vice-versa. Following the analysis of Sec. III B, the $(-)$ branch shows the usual ungapped Bogoliubov behavior: It is phonon-like for $\epsilon_{\mathbf{k}} \ll gn, g_{12}n$, $E^-_{\mathbf{k}} \simeq c\hbar k$ with $c = \sqrt{(g + g_{12})n/2m}$ the sound velocity; It is free-particle-like for $\epsilon_{\mathbf{k}} \gg gn, g_{12}n$, $E^-_{\mathbf{k}} \simeq \epsilon_{\mathbf{k}} + (g + g_{12})n/2$. Conversely, the $(+)$ branch is similar to that of the case with one-body coupling (Sec. III C), although the effective coupling term is renormalized by the two-body interaction strength ($g \to g - g_{12}$). The $(+)$ branch is gapped, and free-particle-like in both low and high-energy limits: For $\epsilon_{\mathbf{k}} \ll (g - g_{12})n, \hbar\Omega_0$, we have $E^+_{\mathbf{k}} \simeq \sqrt{\hbar\Omega_0(\hbar\Omega_0 + (g - g_{12})n)} + \frac{2\hbar\Omega_0 + (g - g_{12})n}{2\sqrt{\hbar\Omega_0(\hbar\Omega_0 + (g - g_{12})n)}}\epsilon_{\mathbf{k}}$; For $\epsilon_{\mathbf{k}} \gg (g - g_{12})n, \hbar\Omega_0$, we have $E^+_{\mathbf{k}} \simeq \epsilon_{\mathbf{k}} + \hbar\Omega_0 + (g - g_{12})n/2$. Therefore, attractive two-body coupling, $g_{12} < 0$, cooperates with one-body coupling to separate the branches: we have $E^-_{\mathbf{k}} < E^+_{\mathbf{k}}$ for any momentum $\mathbf{k}$, and $E^+_{\mathbf{k}} - E^-_{\mathbf{k}}$ increases with both Ω_0 and g_{12}. In

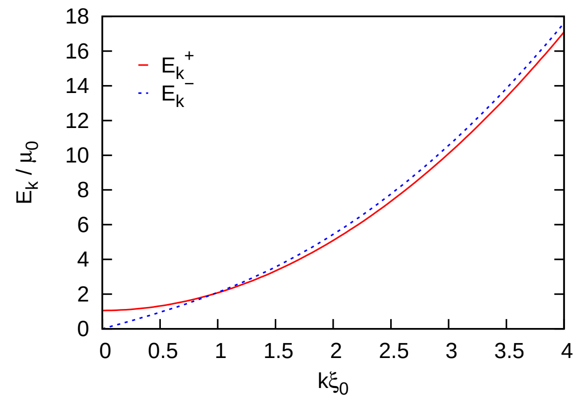

FIG. 6. Bogoliubov spectrum of the coupled excitations in a homogeneous two-component Bose gas with $g_{12} \neq 0$ and $\Omega \neq 0$. Plotted are the two energy branches $E^\pm_{\mathbf{k}}$ [Eqs. (83) and (84)] in the case $g_1 = g_2$ and for $g_{12} = 0.7g_1$ and $\hbar\Omega_0 = 0.4g_1n$. This corresponds to a situation where $g_{12}n > \hbar\Omega_0$ and the $(+)$ and $(-)$ branches cross at a certain momentum k^c (see text). For $g_{12}n < \hbar\Omega_0$, there is no crossing point and the $(+)$ branch is always above the $(-)$ one. Here, $\mu_0 = g_1N/2\mathcal{V}$ is the chemical potential in the absence of any coupling, and $\xi_0 = \hbar/\sqrt{2m\mu_0}$ is the corresponding healing length.

contrast, repulsive two-body coupling, $g_{12} > 0$, competes with one-body coupling, which tends to decrease the separation energy $E^+_{\mathbf{k}} - E^-_{\mathbf{k}}$. For low momentum, the gap in the $(+)$ branch ensures that $E^-_{\mathbf{k}} < E^+_{\mathbf{k}}$. Conversely, for larger momenta, the two branches can exhibit a crossing point if the repulsive interactions are strong enough ($g_{12}n > \hbar\Omega_0$). This happens at the energy $\epsilon^c_{\mathbf{k}} = (\hbar k^c)^2/2m = \hbar\Omega_0[\hbar\Omega_0 + (g - g_{12})n]/2(g_{12}n - \hbar\Omega_0)$.

The computation of the Bogoliubov wavefunctions follows from the same procedure as in the previous sections, which yields

$$f^{\mathrm{m}\pm}_{1\mathbf{k}} = \left[\frac{\epsilon_{\mathbf{k}} + \hbar\Omega_0/2 \pm \hbar\Omega_0/2}{2E^\pm_{\mathbf{k}}}\right]^{1/2} \tag{85}$$

$$f^{\mathrm{p}\pm}_{1\mathbf{k}} = \left[\frac{E^\pm_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} + \hbar\Omega_0 \pm \hbar\Omega_0}\right]^{1/2} \tag{86}$$

$$f^{\mathrm{m}\pm}_{2\mathbf{k}} = \mp\left[\frac{\epsilon_{\mathbf{k}} + \hbar\Omega_0/2 \pm \hbar\Omega_0/2}{2E^\pm_{\mathbf{k}}}\right]^{1/2} \tag{87}$$

$$f^{\mathrm{p}\pm}_{2\mathbf{k}} = \mp\left[\frac{E^\pm_{\mathbf{k}}}{2\epsilon_{\mathbf{k}} + \hbar\Omega_0 \pm \hbar\Omega_0}\right]^{1/2}. \tag{88}$$

Although they are given by the same expressions as in Sec. III C, it should be noticed that the expressions of $E^\pm_{\mathbf{k}}$ have changed according to Eqs. (83) and (84). Their behaviors however remain very similar to the case of Sec. III C. Note that we also recover the particular formulas for $g_{12} = 0$ as well as for $\Omega_0 = 0$ (with branches $(+)$ and $(-)$ being inverted if $g_{12} > 0$).

As in previous the cases, the $f^{\mathrm{m}}_\sigma(\mathbf{r})$ and $f^{\mathrm{p}}_\sigma(\mathbf{r})$ wavefunctions of a single component are always in phase.

The modes associated to the components 1 ($f^{\rm m}_{1\mathbf{k}}, f^{\rm p}_{1\mathbf{k}}$) and 2 ($f^{\rm m}_{2\mathbf{k}}, f^{\rm p}_{2\mathbf{k}}$) are off phase in the (+) branch and in phase in the (−) branch. For attractive interactions $g_{12} < 0$, we have $E^-_{\mathbf{k}} < E^+_{\mathbf{k}}$, so in-phase fluctuations are favored cooperatively by one-body and two-body couplings. Conversely, if the two-body coupling is repulsive and strong enough to compete with the one-body coupling ($g_{12}n > \hbar\Omega_0$), so that the two branches cross, they compete with the following result: For low-energy excitations ($\epsilon_{\mathbf{k}} < \epsilon^c_{\mathbf{k}}$), in-phase fluctuations cost less energy than off-phase fluctuations, whereas it is the opposite for high energy excitations ($\epsilon_{\mathbf{k}} > \epsilon^c_{\mathbf{k}}$).

2. Fluctuations and correlations

Due to the similarity of the dispersion relation and Bogoliubov wavefunctions with the case of Sec. III C, the results for phase and density fluctuations within one component σ are qualitatively identical. In particular, each component is phase coherent in three dimensions but exhibits large phase fluctuations in lower dimensions, driven by the ungapped Bogoliubov-like spectrum of the (−) branch and the corresponding low-momentum divergence of $f^{\rm p-}_{\sigma\mathbf{k}}$.

For the same reason, the correlation function of the relative phase is as well very similar to Sec. III C. It can in particular be rewritten in the form of Eq. (81) with $E^{\pm}_{\mathbf{k}}$ now given by Eqs. (83) and (84). In the case $g_1 = g_2$ we are considering, only the off-phase branch (+) contributes to $G_\theta(\mathbf{r},\mathbf{r}')$. Owing to the gap in the (+) branch, it yields mutual phase coherence in all dimensions. Furthermore, the behavior of $G_\theta(\mathbf{r},\mathbf{r}')$ is identical to that of Sec. III C with a renormalized interaction strength $g \to g - g_{12}$,

$$G_\theta(\mathbf{r},\mathbf{r}') = \frac{1}{n\mathcal{V}}\sum_{\mathbf{k}\neq 0}\left[\sqrt{\frac{\epsilon_{\mathbf{k}}+(g-g_{12})n+\hbar\Omega_0}{\epsilon_{\mathbf{k}}+\hbar\Omega_0}}\coth\left(\frac{E^+_{\mathbf{k}}}{2k_{\rm B}T}\right) -1\right]\times\cos[\mathbf{k}.(\mathbf{r}-\mathbf{r}')]\,. \qquad (89)$$

The relative phase fluctuations therefore decrease when the intensity of the one-body coupling Ω_0 increases, as in the particular case of Sec. III C. We also find that the relative phase fluctuations decrease when the two-body coupling increases, recovering as well the behavior of Sec. III B. It can be seen from Eq. (89) noticing that $\sqrt{(\epsilon_{\mathbf{k}}+(g-g_{12})n+\hbar\Omega_0)/(\epsilon_{\mathbf{k}}+\hbar\Omega_0)} = E^+_{\mathbf{k}}/(\epsilon_{\mathbf{k}}+\hbar\Omega_0)$, that $E^+_{\mathbf{k}}$ decreases when g_{12} increases [see Eq. (84)], and that the function coth is a decreasing function.

Let us discuss the behavior of the relative phase correlation function $G_\theta(\mathbf{r},\mathbf{r}')$ versus temperature. We focus on the one-dimensional case where phase fluctuations are expected to be the more important. The discussion extends the results of Ref. [26] to the case where one-body and two-body coupling coexist. Equation (89) is plotted on Fig. 7 as a function of $|\mathbf{r}-\mathbf{r}'|$ for various temperatures. The function $G_\theta(\mathbf{r},\mathbf{r}')$ increases with the temperature T, as is easily checked from Eq. (89), since the thermal contribution gets more and more important. For $k_{\rm B}T \gg \hbar\Omega_0, (g-g_{12})n$, the quantum contribution can be neglected and $\coth\left(\frac{E^+_{\mathbf{k}}}{2k_{\rm B}T}\right)$ can be safely replaced by $2k_{\rm B}T/E^+_{\mathbf{k}}$ in Eq. (89), yielding the exponentially decaying correlation function

$$G_\theta(\mathbf{r}) = \frac{2mk_{\rm B}T}{n\hbar^2 L_\theta^{-1}}e^{-|\mathbf{r}|/L_\theta}, \qquad (90)$$

where $L_\theta = \sqrt{\frac{\hbar}{2m\Omega_0}}$ is the relative-phase correlation length [26]. This approximate formula still holds for smaller values of the temperature in the large separation limit, predicting thus a correct correlation length. The latter then weakly depends on the two-body coupling and decreases when the one-body coupling increases.

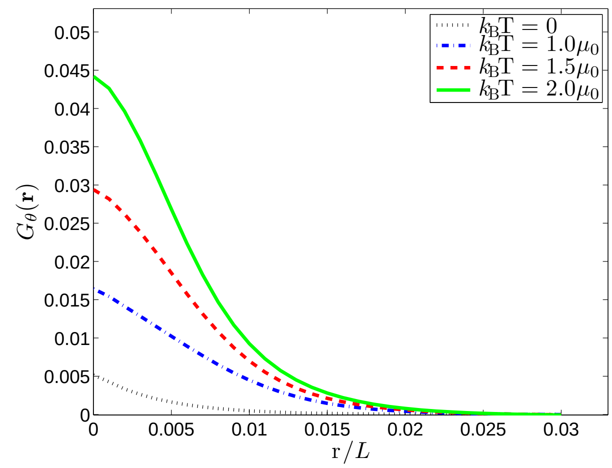

FIG. 7. Correlation function of the relative phase for a two-component Bose gas with one-body ($\Omega_0 \neq 0$) and two-body ($g_{12} \neq 0$) couplings, plotted for various temperatures ($k_{\rm B}T/\mu_0 = 0,\ 1,\ 1.5,\ 2$) in the case where $g_1 = g_2 \equiv g$. The parameters here correspond to $N = 10^4$ atoms of ^{87}Rb ($m \simeq 144 \times 10^{-27}$kg) in a 1D box of size $2L = 10^{-4}$m, and interacting via the scattering length $a_1 = a_2 = 5.95$nm. It corresponds in the absence of any coupling to the chemical potential $\mu_0 = gn = 7.88 \times 10^{-31}$J, which we choose as the energy unit. In these units, we use the parameters $\hbar\Omega_0 = 1\mu_0$ and $g_{12}n = 0.75\mu_0$.

We finally discuss the relative phase fluctuations, which are given by $G_\theta(\mathbf{r}=0)$. As already pointed out, the relative phase fluctuations always decrease with the one-body coupling Ω_0, which thus favors mutual phase coherence between the two condensates. Moreover, repulsive two-body coupling ($g_{12} > 0$) tends to reduce the fluctuations of the relative phase while attractive two-body coupling enhance them. The temperature dependence of those fluctuations is shown in Fig. 8. The zero-temperature fluctuations, which are given by their quantum contribution, are smaller than those of a sin-

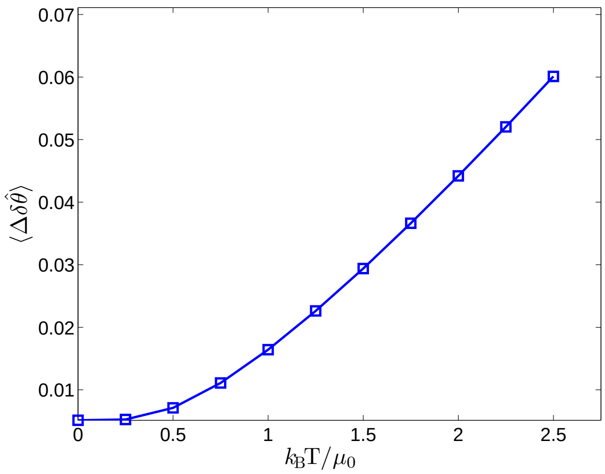

FIG. 8. Relative phase fluctuations as a function of temperature for a two-component Bose gas with one-body ($\Omega_0 \neq 0$) and two-body ($g_{12} \neq 0$) couplings, plotted for the same parameters as in Fig. 7. While the quantum fluctuations are small, the thermal contribution increases with temperature. Note that the temperatures considered here are outside the range of validity of Eq. (90).

gle condensate [26]. The fluctuations then unsurprisingly increase with temperature. The linear dependence predicted by Eq. (90), $G_\theta(\mathbf{r}=0) \simeq \frac{2mk_BT}{n\hbar^2 L_\theta^{-1}}$, is expected only in the regime where $k_BT \gg \hbar\Omega_0, (g-g_{12})n$, which is beyond the range of temperatures considered here. In addition, the fluctuations remain small provided $k_BT \ll n\hbar\sqrt{\hbar\Omega_0/m}$. Hence, the one-body coupling favors local mutual phase coherence between the two components.

IV. CONCLUSIONS

In this paper, we have derived a general meanfield theory for a two-component Bose gas, in the presence of both one-body and two-body couplings. We considered the most general situation where both one-body and two-body couplings can be position dependant, and where the gas can experience a component-dependent external potential. Our formulation uses the phase-density formalism, which allows us to capture both cases of true condensates and quasi-condensates with large phase fluctuations. We have written the coupled Gross-Pitaevskii equations, which determine the ground-state background, as well as the Bogoliubov equations, which determine the pair-excitation spectrum of the mixture. We obtained general formulas for phase and density correlation functions within each component, as well as for their relative phase, at zero and finite temperature.

We have then applied our formalism to homogeneous cases with only two-body coupling (Sec. III B), only one-body coupling (Sec. III C), or both one-body and two-body couplings (Sec. III D). Our discussion then focused on the excitation spectrum and the relative phase fluctuations. We summarize our main results in the following.

Two-body coupling (Sec. III B) — When only the two-body coupling is present, the excitation spectrum exhibits two gapless Bogoliubov-like branches. These two branches correspond to in-phase and off-phase fluctuations of the two components. They are separated by an energy-dependent quantity that converges to $\Delta = |g_{12}|(n_1+n_2)$ in the high-energy limit. As regards phase and density fluctuations, each component behaves as an effective single-component Bose gas with coupling parameters that are renormalized by the inter-species two-body coupling. This is also the case of the relative phase since the two-body coupling does not constraint it. Nevertheless, the relative-phase fluctuations and correlations are mostly determined by the off-phase branch of the spectrum, provided that the intra-species interaction strengths are not too different. The influence of the two-body coupling on the relative phase fluctuations was discussed in the case where the intra-species interaction strengths are equal ($g_1 = g_2$). One one hand, an increasing g_{12} tends to lower the contributing off-phase branch, hence increasing its thermal occupancy. On the other hand, it enhances the amplitude of off-phase density fluctuations, and therefore reduces the amplitude of phase fluctuations in the contributing off-phase branch. We found that the latter effect always dominates. Therefore, repulsive inter-component interactions suppress relative phase fluctuations while attractive inter-component interactions enhance relative phase fluctuations.

One-body coupling (Sec. III C) — The one-body coupling, and in particular its phase, imposes the relative phase of the two components at the meanfield level. Then, the fluctuations of the relative phase only depend on the modulus of the one-body coupling. The two branches of the excitation spectrum are different from the previous case. While the lower branch is of the Bogoliubov type and corresponds to in-phase relative fluctuations, the upper branch is gapped and corresponds to off-phase relative fluctuations. In the case where the intra-species interaction terms of the two components are equal, the relative phase fluctuations are strictly governed by the off-phase branch. The gap then cuts the low-energy divergence of the corresponding excitation functions, and the relative phase fluctuations are suppressed. Hence one-body coupling suppresses the fluctuations of the relative phase, independently of its sign.

One-body and two-body couplings (Sec. III D) — The general case, where both one-body and two-body couplings are present, combines the behaviors found in the two previous cases. The spectrum is again made of two branches. The first branch, which corresponds to in-phase fluctuations of the two Bose gases, is of the Bogoliubov type. It depends only on the two-body coupling while being unaffected by one-body coupling. The second branch, which corresponds to off-phase fluctuations, is gapped. The two branches cross each other at a given momentum if the two-body coupling is repulsive and exceeds the one-body coupling. Here again the relative phase is imposed by the one-body coupling at the meanfield level and the fluctuations depend only on its modulus. As in the case where two-body coupling is absent, one-body coupling always favors relative-phase coherence of the two Bose gases. Then, repulsive two-body coupling cooperates with one-body coupling and further suppresses relative-phase fluctuations, while attractive two-body coupling competes with one-body coupling and enhances relative-phase fluctuations. However the correlation length of the relative phase decreases when the one-body coupling increases.

Our work generalizes previous results to the case where both one-body and two-body couplings are present between the two Bose components. The homogeneous cases we have analyzed are expected to contain the main physics of relative-phase coherence. The formalism that we have developed here can be directly applied to more complicated situations. For instance, the effect of inhomogeneous trapping, which can be component-dependent, is particularly relevant in the context of ultracold-atom systems. In this case, one may resort to numerical solutions of the Gross-Pitaevskii and Bogoliubov equations. Other interesting applications of this formalism include the study of the effects of strong inhomogeneities in interacting Bose gases, in particular random couplings, which is attracting much attention in ultracold-atom systems [71]. One may envision several applications. First, disordered potential have been shown to induce Anderson localization of the Bogoliubov excitations in single-component Bose gases [72–75]. How does it extend to the case of coupled Bose gases ? Second, disorder can be included in interaction terms using inhomogeneous Feshbach resonances [76]. What would be the effect of random inter-species coupling ? Third, disorder can be included in one-body coupling, which has been shown to produce random-field-induced-order of the relative phase of two Bose-Einstein condensate at zero temperature [35, 36, 77, 78]. How do finite temperature affect this behavior ?

Note added. While completing this manuscript, we were made aware of a related work, reporting analysis of the excitation spectrum and the structure factors of coupled two-component Bose-Einstein condensates [79].

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Appendix A: General case with one-body ($\Omega \neq 0$) and two-body ($g_{12} < g_1, g_2$) couplings

We compute here the excitation spectrum of the two-component Bose gas in the general case where both one-body and two-body couplings are present and the intra-component couplings g_1 and g_2 can be different. Following the same approach as in Secs. III B, III C and III D, we rewrite the Bogoliubov equations in terms of the $f^{\mathrm{p,m}}_{\sigma\mathbf{k}}$ functions :

$$E_{\mathbf{k}} f^{\mathrm{m}}_{\sigma\mathbf{k}} = \left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}}\right) f^{\mathrm{p}}_{\sigma\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{\bar\sigma\mathbf{k}}$$

$$E_{\mathbf{k}} f^{\mathrm{p}}_{\sigma\mathbf{k}} = \left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}} + 2g_\sigma n_\sigma\right) f^{\mathrm{m}}_{\sigma\mathbf{k}} + \left(2g_{12}\sqrt{n_1 n_2} - \frac{\hbar\Omega_0}{2}\right) f^{\mathrm{m}}_{\bar\sigma\mathbf{k}}$$

where $\bar\sigma$ is the conjugate of component σ. Using the normalization condition $f^+_{1\mathbf{k}} f^-_{1\mathbf{k}} + f^+_{2\mathbf{k}} f^-_{2\mathbf{k}} = 1$, it yields :

$$E^2_{\mathbf{k}} f^{\mathrm{p}}_{\sigma\mathbf{k}} = \left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}} + 2g_\sigma n_\sigma\right)\left[\left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}}\right) f^{\mathrm{p}}_{\sigma\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{\bar\sigma\mathbf{k}}\right] + \left(2g_{12}\sqrt{n_1 n_2} - \frac{\hbar\Omega_0}{2}\right)\left[-\frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{\sigma\mathbf{k}} + \left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_\sigma}{n_{\bar\sigma}}}\right) f^{\mathrm{p}}_{\bar\sigma\mathbf{k}}\right] \tag{A1}$$

$$E_{\mathbf{k}} = f^{\mathrm{p}}_{1\mathbf{k}}\left[\left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_2}{n_1}}\right) f^{\mathrm{p}}_{1\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{2\mathbf{k}}\right] + f^{\mathrm{p}}_{2\mathbf{k}}\left[\left(\epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_1}{n_2}}\right) f^{\mathrm{p}}_{2\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{1\mathbf{k}}\right]. \tag{A2}$$

In order to simplify the notations, let us define $\epsilon_{\sigma\mathbf{k}} = \epsilon_{\mathbf{k}} + \frac{\hbar\Omega_0}{2}\sqrt{\frac{n_{\bar\sigma}}{n_\sigma}}$, $U_\sigma = g_\sigma n_\sigma$, and $U_{12} = g_{12}\sqrt{n_1 n_2}$. Then, we have $A_{\mathbf{k}\sigma} = \epsilon_{\sigma\mathbf{k}}(\epsilon_{\sigma\mathbf{k}} + 2U_\sigma) - \frac{\hbar\Omega_0}{2}\left(2U_{12} - \frac{\hbar\Omega_0}{2}\right)$ and $B_{\mathbf{k}\sigma} = \epsilon_{\bar\sigma\mathbf{k}}\left(2U_{12} - \frac{\hbar\Omega_0}{2}\right) - \frac{\hbar\Omega_0}{2}(\epsilon_{\sigma\mathbf{k}} + 2U_\sigma)$. With these notations, Eqs. (A1) and (A2) write

$$f^{\mathrm{p}}_{\bar\sigma\mathbf{k}} B_{\mathbf{k}\sigma} = f^{\mathrm{p}}_{\sigma\mathbf{k}}[E^2_{\mathbf{k}} - A_{\mathbf{k}\sigma}]$$

$$E_{\mathbf{k}} = f^{\mathrm{p}}_{1\mathbf{k}}[\epsilon_{1\mathbf{k}} f^{\mathrm{p}}_{1\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{2\mathbf{k}}] + f^{\mathrm{p}}_{2\mathbf{k}}[\epsilon_{2\mathbf{k}} f^{\mathrm{p}}_{2\mathbf{k}} - \frac{\hbar\Omega_0}{2} f^{\mathrm{p}}_{1\mathbf{k}}].$$

We thus find the excitation spectrum

$$E_{\mathbf{k}}^{\pm}=\sqrt{\frac{1}{2}(A_{\mathbf{k}1}+A_{\mathbf{k}2})\pm\sqrt{(A_{\mathbf{k}1}-A_{\mathbf{k}2})^2/4+B_{\mathbf{k}1}B_{\mathbf{k}2}}}. \tag{A3}$$

In the particular case where $\Omega_0=0$, $A_{\mathbf{k}\sigma}=\epsilon_{\mathbf{k}}(\epsilon_{\mathbf{k}}+2g_\sigma n_\sigma)$ and $B_{\mathbf{k}\sigma}=\epsilon_{\mathbf{k}}2g_{12}\sqrt{n_1n_2}$, so that we recover the result of Sec. III B. In the case where both one-body and two-body couplings are present and $n_1=n_2$, $A_{\mathbf{k}1}=A_{\mathbf{k}2}\equiv A_{\mathbf{k}}=\left(\epsilon_{\mathbf{k}}+\frac{\hbar\Omega_0}{2}\right)\left(\epsilon_{\mathbf{k}}+\frac{\hbar\Omega_0}{2}+gn\right)-\frac{\hbar\Omega_0}{2}\left(ng_{12}-\frac{\hbar\Omega_0}{2}\right)$ and $B_{\mathbf{k}1}=B_{\mathbf{k}2}\equiv B_{\mathbf{k}}=\left(\epsilon_{\mathbf{k}}+\frac{\hbar\Omega_0}{2}\right)\left(ng_{12}-\frac{\hbar\Omega_0}{2}\right)-\frac{\hbar\Omega_0}{2}\left(\epsilon_{\mathbf{k}}+\frac{\hbar\Omega_0}{2}+gn\right)$, and we recover the spectrum of Sec. III D, with possible inversion of the two branches depending on the sign of $B_{\mathbf{k}}$.

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