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Homodyne estimation of Gaussian quantum discord

Rémi Blandino,[1,*] Marco G. Genoni,[2,†] Jean Etesse,[1] Marco Barbieri,[1]
Matteo G.A. Paris,[3,4] Philippe Grangier,[1] and Rosa Tualle-Brouri[5,1]

[1]*Laboratoire Charles Fabry, Institut d'Optique, CNRS, Université Paris-Sud, Campus Polytechnique, RD 128, 91127 Palaiseau cedex, France*
[2]*QOLS, Blackett Laboratory, Imperial College London, London SW7 2BW, UK*
[3]*Dipartimento di Fisica, Universitá degli Studi di Milano, I-20133, Milano, Italy*
[4]*CNISM – Udr Milano, I- 20133, Milano, Italy*
[5]*Institut Universitaire de France, 103 boulevard St. Michel, 75005, Paris, France*

We address the experimental estimation of Gaussian quantum discord for two-mode squeezed state, and demonstrate a measurement scheme based on a pair of homodyne detectors assisted by Bayesian analysis. Our scheme provides nearly optimal estimation for small value of discord, where Bayesian analysis allows to greatly improves performances. Although homodyne detection is not optimal for Gaussian discord, the noise ratio to the ultimate quantum limit is limited to about 10 dB. Our results illustrate how suitable data processing can decrease significantly the uncertainty when optimal detection schemes are not available.

Quantum correlations are thought to be central resources for quantum technology. These tight connections empower the advantages shown by the exploitation of quantum coding in applications to cryptography, computation and sensing. While at first entanglement was recognized to be the unique form of quantum correlations, novel concepts have been introduced to capture either more specific aspects, such as quantum steering [1, 2], or, to the other end of the spectrum, more general occurrences. Quantum discord represents the most successful attempt to broaden the understanding of the nature of quantum correlations [3, 4]. It has recently attracted considerable attention, due to its possible usefulness as a resource in mixed-state quantum computing [5, 6], where entanglement is shown to be exponentially small [7, 8]. Furthermore, it has been demonstrated that discord does play a role in the activation of multipartite entanglement [9], entanglement generation by measurement [10], state merging [11], and for complete positivity of evolutions [12, 13].

Direct experimental observation of quantum discord has been undertaken either by direct inspection of the full density matrix [14], or by using a witness [15, 16] with no concern about the optimality of the scheme. A key challenge is then to find strategies for the quantification of this resource with precision, and to understand its fundamental limit to the precision of the estimation procedure by introducing proper Cramér-Rao bounds (CRB) [17–19]. This has already been done for the entanglement [20] and optimal estimators have been experimentally shown to attain the quantum limit for different families of qubit states [21]. From the perspective of quantum metrology, this is highly nontrivial, since there exists no observable directly related to quantum discord. A proper estimator is then needed, which might depend on several characteristic parameters of the quantum state. In such a multiparameter problem, finding an optimized detection scheme might be hard, and could demand complex experimental apparata or heavy post-processing of the data.

In this Letter, we demonstrate homodyne estimation of Gaussian quantum discord in continuous variable systems [22, 23], and compare the achieved level of precision with the classical CRB for homodyne detection, and with the quantum CRB, which sets the ultimate precision allowed by quantum mechanics. Our findings show how a suitable Bayesian data processing may significantly improve precision, especially in the estimation of small values of discord. More generally, our results illustrate how a suitable data processing can decrease significatively the uncertainty when the optimal detection scheme is not available.

Our investigation is concerned with an important class of Gaussian states, i.e. the two-mode squeezed thermal states (STS) naturally produced by a non-collinear optical parametric amplifier (OPA). If we introduce the two-mode squeezing operator $S_2(s) = \exp\left(s(a_0^\dagger a_1^\dagger - a_0 a_1)\right)$, and the thermal state $\nu(N) = \frac{1}{N+1}\sum_n (\frac{N}{N+1})^n |n\rangle\langle n|$, we can write the STS as

$$\varrho(N_s, N_t) = S_2(s)\nu(N_t) \otimes \nu(N_t) S_2(s)^\dagger, \qquad (1)$$

and thus can be fully described by the two parameters $N_s = \sinh^2 s$ and N_t, representing, respectively, the effective amount of *squeezing photons* and *thermal photons*. In fact, spurious effects such as unwanted amplification, result in a loss of purity of the squeezed state by thermalisation, but do not affect the Gaussian character of the emission, so the form of the density matrix (1) provides a fully general description of the output of a realistic OPA.

The study of quantum discord for continuous variable systems has been addressed so far uniquely for Gaussian states and measurements; there exists a general closed formula for discord given an arbitrary covariance matrix [23], but in the following we will use a simplified expression holding for a particular class to which STS belong [22]. We can estimate the discord from N_s and N_t as we varied the pump power of our OPA [24]. For each power setting, these two parameters are extracted by the outcome of two homodyne detectors, one on each mode, which measure pairs of quadratures $\{X_0, X_1\}$ and $\{P_0, P_1\}$ (Fig.1). From these, we can evaluate the four linear combinations

$$Q^{(1,2)} = \frac{X_0 \pm X_1}{\sqrt{2}} \qquad Q^{(3,4)} = \frac{P_0 \pm P_1}{\sqrt{2}} \qquad (2)$$

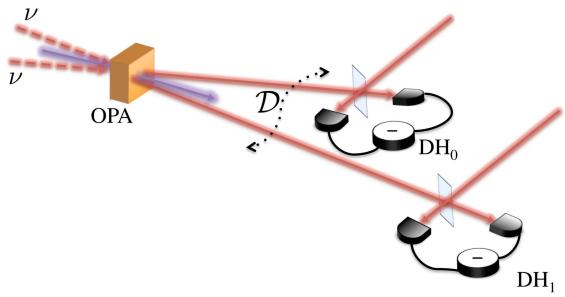

FIG. 1: Conceptual layout of the discord estimation. Our non-collinear OPA is based on a nonlinear $KNbO_3$ crystal, pumped by a frequency-doubled Ti:Sapph laser (repetition rate 800kHz, wavelength λ_p=425nm, pulse duration 120fs). This produces a STS with discord D depending on the pump power, i.e. on their average photon number. the two mode are measured by two homodyne detectors DH_0 and DH_1. The relative phase between the local oscillators is locked by mapping them to different polarizations on the same spatial mode. In this way, we can record blocks of 20000 values of pairs of quadratures for $\{X_0, X_1\}$ and the same for $\{P_0, P_1\}$.

where $Q^{(1)}$ and $Q^{(4)}$ are squeezed quadratures, while $Q^{(2)}$ and $Q^{(3)}$ are anti-squeezed; in particular M_q measurement outcomes are recorded for each one of the four quadratures. The corresponding variances, $\sigma^2(Q_{\rm sq})$ and $\sigma^2(Q_{\rm asq})$, that can be obtained from the experimental data, can be rewritten as function of N_s and N_t as follows

$$\sigma^2(Q_{\rm sq/asq}) = (1 + 2N_s \mp 2\sqrt{N_s(1+N_s)})(1+2N_t) \quad (3)$$

The expressions obtained can be then inverted to obtain the experimental estimate $N_s^{\rm inv}$ and $N_t^{\rm inv}$, along with the relative uncertainties $\sigma^2(N_s^{\rm inv})$ and $\sigma^2(N_t^{\rm inv})$. These values can be used in the expression for discord [24] to calculate its value $D^{\rm inv}$, and the uncertainty $\sigma^2(D^{\rm inv})$ by simple error propagation.

One can use the same data and refine the estimation by using a Bayesian analysis. As described above, each data sample corresponds to $M_q = 2 \cdot 10^4$ measurement of each of the four quadratures. The total sample, thus correspond to $M_T = 4M_q$ homodyne outcomes

$$\mathcal{X} = \{q_1^{(1)}, .., q_{M_q}^{(1)}, q_1^{(2)}, .., q_{M_q}^{(2)}, q_1^{(3)}, .., q_{M_q}^{(3)}, q_1^{(4)}, .., q_{M_q}^{(4)}\}.$$

The overall sample probability can be evaluated as

$$p(\mathcal{X}|N_s, N_t) = \prod_{k=1}^{4}\prod_{j=1}^{M_q} p_k(q_j^{(k)}|N_s, N_t) \quad (4)$$

where the probability of obtaining the outcome $q_j^{(k)}$ by measuring the quadrature $Q^{(k)}$ is a Gaussian distribution

$$p_k(q_j^{(k)}|N_s, N_t) = \frac{1}{\sqrt{2\pi\sigma_k^2}} \exp\left(-\frac{(q_j^{(k)})^2}{2\sigma_k^2}\right). \quad (5)$$

For squeezed quadratures ($k = \{1, 4\}$) we substitute $\sigma_k^2 = \sigma^2(Q_{\rm sq})$, while for anti-squeezed quadrature ($k = \{2, 3\}$) $\sigma_k^2 = \sigma^2(Q_{\rm asq})$. By means of the Bayes theorem, we obtain the *a-posteriori* probability

$$p(N_s, N_t|\mathcal{X}) = \frac{1}{\mathcal{N}} p(\mathcal{X}|N_s, N_t)p_0(N_s)p_0(N_t), \quad (6)$$

$$\mathcal{N} = \int dN_s\, dN_t\, p(\mathcal{X}|N_s, N_t)p_0(N_s)p_0(N_t). \quad (7)$$

where the $p_0(N_s)$ and $p_0(N_t)$ are the so-called *a-priori* probability distributions for the two parameters. In our procedure, we use the results of the inversion estimation to construct these *a-priori* distributions. That is, we consider $p_0(N_s)$ and $p_0(N_t)$ as Gaussian functions with respectively, mean values equal to $N_s^{\rm inv}$ and $N_t^{\rm inv}$, and variances equal to $\sigma^2(N_s^{\rm inv})$ and $\sigma^2(N_t^{\rm inv})$. Then, we can use the *a-posteriori* probability distribution evaluated as in Eq. (6) to obtain an estimate of the two parameters and of their variances. Mathematically,

$$N_s^{\rm bay} = \int dN_s\, dN_t\, N_s\, p(N_s, N_t|\mathcal{X}) \quad (8)$$

$$\sigma^2(N_s^{\rm bay}) = \int dN_s\, dN_t\, (N_s - N_s^{\rm bay})^2\, p(N_s, N_t|\mathcal{X}) \quad (9)$$

$$N_t^{\rm bay} = \int dN_s\, dN_t\, N_t\, p(N_s, N_t|\mathcal{X}) \quad (10)$$

$$\sigma^2(N_t^{\rm bay}) = \int dN_s\, dN_t\, (N_t - N_t^{\rm bay})^2\, p(N_s, N_t|\mathcal{X}). \quad (11)$$

Again, by using the formula of the discord for two-mode STS in [24] and by propagating the errors, we obtain an estimate $D^{\rm bay}$ for the discord, alongs with its variance $\sigma^2(D^{\rm bay})$.

The value of discord depends on both the squeezing and thermal photons. Consequently, its estimation is inherently a multi-parameter problem, and we have to identify the relevant physical parameters to evaluate the correct CRB. In the multiparameter scenario, the quantum Fisher information (QFI) associated to a vector of parameters $\bar{\lambda}=\{\lambda_i\}_{0\le i\le n}$ is in the form of a matrix $\boldsymbol{H}$. This sets a lower bound on the covariance $\sigma_{ij}^2=\langle\lambda_i\lambda_j\rangle-\langle\lambda_i\rangle\langle\lambda_j\rangle$ after M repetitions on the experiment:

$$\sigma_{ij}^2 \ge \frac{1}{M}\left(\boldsymbol{H}^{-1}\right)_{ij} \quad (12)$$

In the specific case of our experiment, we can bound the uncertainty on the discord D of the states we prepare as: $\sigma^2(D) \ge \frac{1}{M}(\boldsymbol{H}^{-1})_{DD}$. While our measurement strategy has the advantage of being simple, it is not expected to saturate the quantum CRB. We therefore need to compare it to the classical CRB associated to our specific measurement, which is analogously described by a classical Fisher information (FI) matrix $\boldsymbol{F}$.

In the evaluation of the correct bound, we need a suitable parametrization of the state, so that in the expression (12) one parameter only actually varies, while the others are kept fixed: this can not be the case for the number of thermal and squeezing photons, as both of them change with the pump power.

Therefore, we need to reshape the QFI matrices for a different pair of parameters, so to consider those which are more directly connected to the experimental conditions. We start by considering the first couple $\boldsymbol{\lambda}_1 = \{N_s, N_t\}$; by using the formulas described in [24], we obtain

$$\boldsymbol{H}^{(1)} = \mathrm{diag}\left(\frac{(1+2N_t)^2}{N_s(1+N_s)(1+2N_t+2N_t^2)}, \frac{1}{N_t(1+N_t}\right). \tag{13}$$

As explained above, thermal photons appear because of imperfections in the operation of the OPA and because of loss. We can then reparametrize our state by taking into consideration the effective squeezing strength r, and a parasite amplification with strength γr [25, 26]. The overall homodyne detection can be separately calibrated, obtaining $\eta = 0.62$. Thus we can rewrite the matrix (13) in terms of the two unknown physical parameters $\boldsymbol{\lambda}_2 = \{r, \gamma\}$ via the expression $\boldsymbol{H}^{(2)} = B_{12}\boldsymbol{H}^{(1)}B_{12}^T$, where B_{12} is the transfer matrix for this change of variables [24]. Next, since the physical parameter that changes during our experiment, resulting in the variation of the amount of discord, is the squeezing parameter r (while γ and η can be considered to remain constant), we perform the last change of variable, by considering $\boldsymbol{\lambda}_3 = \{D, \gamma\}$. Again the QFI matrix can be obtained as $\boldsymbol{H}^{(3)} = B_{23}\boldsymbol{H}^{(2)}B_{23}^T$, and the bound on the variance for the quantum discord can be easily evaluated as described in Eq. (12).

We also want to derive the classical CRB for quantum discord, that we obtain if we consider as measurement homodyne detection of squeezed and anti-squeezed quadratures of a two-mode squeezed thermal state. Let us start by considering the Fisher information matrix we obtain if we want to estimate the two parameters $\boldsymbol{\lambda}_1 = \{N_s, N_t\}$ by means of homodyne detection on a certain quadrature Q_ϕ. Since the state is Gaussian, the conditional probability distribution of measuring a value x, is a Gaussian function, with zero mean, and variance $\boldsymbol{\sigma}^2(Q_\phi)$. By using the formulas in [24] and evaluating some Gaussian integrals, one easily obtains the following formula for the Fisher matrix elements

$$\boldsymbol{F}_{\mu\nu} = \frac{1}{2\sigma^2(Q_\phi)}\frac{\partial\sigma^2(Q_\phi)}{\partial\lambda_\mu}\frac{\partial\sigma^2(Q_\phi)}{\partial\lambda_\nu} \tag{14}$$

where $\lambda_\mu = \{N_s, N_t\}$. If one considers to measure the squeezed or the anti-squeezed quadratures one obtains the following FI matrices:

$$\boldsymbol{F}^{\mathrm{sq/asq}} = \begin{pmatrix} \frac{1}{2N_s+2N_s^2} & \mp\frac{1}{\sqrt{N_s(1+N_s)(1+2N_t)}} \\ \mp\frac{1}{\sqrt{N_s(1+N_s)(1+2N_t)}} & \frac{2}{(1+2N_t)^2} \end{pmatrix}$$

If we perform a fixed number of measurements, where half of them are done on the squeezed quadratures, and the remaining ones on the anti-squeezed quadratures, the overall FI matrix which will give the CRB for the two parameters $\boldsymbol{\lambda}_1 = \{N_s, N_t\}$ is obtained as

$$\boldsymbol{F}^{(1)} = \frac{\boldsymbol{F}^{\mathrm{sq}} + \boldsymbol{F}^{\mathrm{asq}}}{2} = \begin{pmatrix} \frac{1}{2N_s+2N_s^2} & 0 \\ 0 & \frac{2}{(1+2N_t)^2} \end{pmatrix}. \tag{15}$$

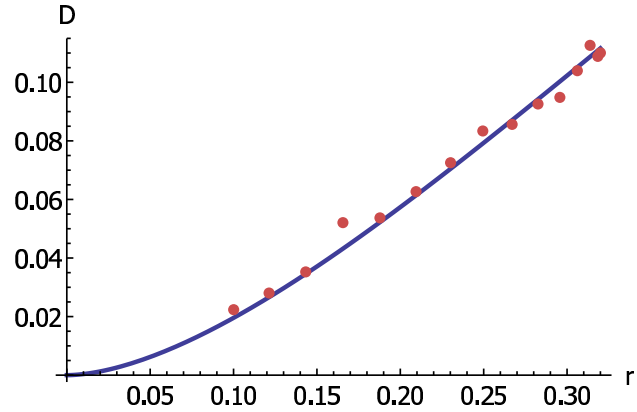

FIG. 2: Experimental values of Gaussian quantum discord from homodyne data and Bayesian estimation. The points correspond to the estimated experimental values, while the solid line is the theoretical prediction for η=0.62 and γ=0.58 (the value of γ has been extracted from a best-fit of the points).

To obtain the CRB for the quantum discord, we can proceed as we showed for the quantum CRB, simply replacing the QFI matrices, with the FI ones. The values of the discord obtained using our Bayesian estimation are shown in Fig. 2: the points indicate the experimental data, while the solid line describes the model (1), where the homodyne efficiency η and the relative parasite gain γ are kept to a constant value. Our model is in excellent agreement with the data, so we can be confident of that the CRB calculated after the matrix (13) reliably describes the ultimate limit for precision.

In the formulae of the Bayes rule (6), we need to multiply several probabilities (5), which rapidly give a number hardly manageable by reasonable computing power: this sets a limit to the number of quadrature values one can effectively use to about 800 points. In order to use larger samples, we have divided our data in $N_b = 10^2$ blocks of 200 points for each of the quadratures (2), calculated the Bayesian estimation of the discord for each block, then considered the average weighted on the associated uncertainties. We notice that the *a priori* probabilities (5) are calculated from the whole set of data containing M_T values: as they intervene in the evaluation for each block, the overall number of resources to be considered is $M = N_b \cdot M_T$.

The comparison between our experimental uncertainties and both the Cramér-Rao limit for our detection (14) and the quantum Cramér-Rao limit is shown in Fig.3, where we report the quantity $K_M = M\sigma^2(D)/(\boldsymbol{F}^{-1})_{DD}$(or the analogous quantity involving the QFI) expressed in dB. K_M is the variance of the discord estimator from homodyne data multiplied by the number of resources and divided by the relevant elements of the (quantum) inverse Fisher matrix. For K_M

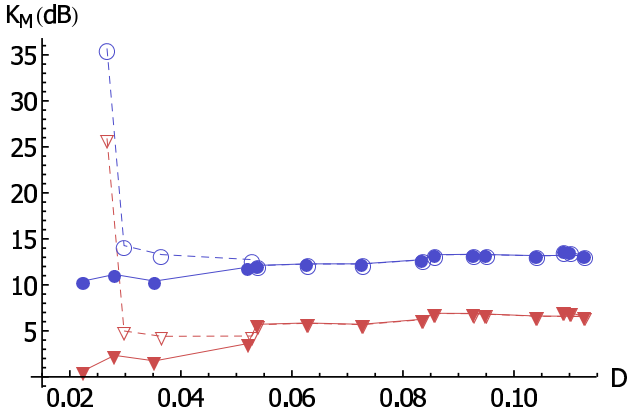

FIG. 3: The noise ratio K_M as a function of discord. Circles and triangles correspond respectively to the quantum and the classical CRB. Solid points refer to Bayesian estimation, while empty ones correspond to estimation by inversion. Notice that the number of resources for Bayesian estimation is $M = N_b \cdot M_T$, while for the inversion method $M = M_T$.

equal to unity we have optimal estimation. Solid points refer to Bayesian estimation while empty ones correspond to estimation by inversion. We notice that for low values of discord, the Bayesian technique provides a nearly optimal estimator for the chosen measurement strategy. Also, we notice that in this region estimation by inversion is much noisier. For increasing values, the observed variances depart from the optimum by less than an order of magnitude: as Bayesian estimation rapidly converges to optimal, we can attribute this trend to actual variations of the value of the discord in the experiment, becoming more important than statistical fluctuations when the discord increases.

The measurement we have adopted has the considerable advantage of being the simplest experimental option; however, simplicity always comes at a price, and we do not expect it to deliver the best estimator for discord as established by the quantum CRB. In the limit of low discord, we measure a ratio of about 10 dB , which tells us that the price we have to pay is quite reasonable. The departure from the quantum CRB then slightly increases with discord.

In conclusion, we have presented the first experimental estimation of Gaussian quantum discord for two-mode squeezed state. Our scheme is based on homodyne detection assisted by Bayesian analysis. Our results are in excellent agreement with the theoretical model, and this allows us to perform a reliable precision analysis. We found that homodyne estimation shows about 10 dB of added noise compared to the ultimate bound imposed by the quantum Fisher information, with Bayesian analysis that improves performances for small values of discord. We have also compared our results with the CRB for homodyne detection and found that the estimation is nearly optimal for small values of discord, where Bayesian estimation enhances precision significantly compared to simple data inversion. The usefulness of quantum discord as a resource for quantum technology is a heavily debated topic, and a definitive answer may only come from experiments involving carefully prepared quantum states. Our results contribute to the precise characterization of Gaussian discord and illustrate how a suitable data processing can decrease significatively the uncertainty when optimal detection schemes are not available.

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* Electronic address: remi.blandino@institutoptique.fr
† Electronic address: m.genoni@imperial.ac.uk

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SUPPLEMENTAL MATERIAL

Multiparametric quantum estimation

Here we present the case where the estimation of more than one parameter has to be performed. We define a family of quantum states $\varrho_{\boldsymbol{\lambda}}$ which depends on a set of N parameters $\boldsymbol{\lambda} = \{\lambda_\mu\}$, $\mu = 1, \dots, N$. In this case the geometry of the estimation problem is contained in the QFI matrix, whose elements are defined as

$$\boldsymbol{H}(\boldsymbol{\lambda})_{\mu\nu} = \mathrm{Tr}\left[\varrho_{\boldsymbol{\lambda}} \frac{L_\mu L_\nu + L_\nu L_\mu}{2}\right], \tag{16}$$

and where we have introduce the Symmetric Logarithmic Derivatives (SLD) L_μ corresponding to the parameter λ_μ, as the selfadjoint operator that satisfies the equation

$$\frac{L_\mu \varrho_{\boldsymbol{\lambda}} + \varrho_{\boldsymbol{\lambda}} L_\mu}{2} = \frac{\partial \varrho_{\boldsymbol{\lambda}}}{\partial \lambda_\mu}. \tag{17}$$

In terms of eigenvalues and eigenvectors of $\varrho_{\boldsymbol{\lambda}}$, by denoting with ∂_μ the partial derivative respect to λ_μ, we have

$$\begin{aligned}\boldsymbol{H}(\boldsymbol{\lambda})_{\mu\nu} = &\sum_n \frac{(\partial_\mu a_n)(\partial_\nu a_n)}{a_n} + \sum_{n\neq m} \frac{(a_n - a_m)^2}{a_n + a_m} \times \\ &\times (\langle\psi_n|\partial_\mu\psi_m\rangle\langle\partial_\nu\psi_m|\psi_n\rangle + \langle\psi_n|\partial_\nu\psi_m\rangle\langle\partial_\mu\psi_m|\psi_n\rangle).\end{aligned} \tag{18}$$

The QFI matrix here defined provides a lower bound (the quantum Cramér-Rao bound) on the covariance matrix $\boldsymbol{\gamma}_{\mu\nu} = \langle\lambda_\mu\lambda_\nu\rangle - \langle\lambda_\mu\rangle\langle\lambda_\nu\rangle$, *i.e.*,

$$\boldsymbol{\gamma} \geq \frac{1}{M}\boldsymbol{H}(\boldsymbol{\lambda})^{-1}. \tag{19}$$

In the multiparametric case this bound is not in general achievable, on the other hand, the diagonal elements of the inverse Fisher matrix provide achievable bounds for the variances of single parameter estimators, at fixed value of the others

$$\mathrm{Var}(\lambda_\mu) = \boldsymbol{\gamma}_{\mu\mu} \geq \frac{1}{M}\boldsymbol{H}(\boldsymbol{\lambda})^{-1}_{\mu\mu}. \tag{20}$$

Let us now suppose that we are interested in the estimation of different set of parameters $\tilde{\boldsymbol{\lambda}} = \{\tilde{\lambda}_\nu = \tilde{\lambda}_\nu(\boldsymbol{\lambda})\}$ which are functions of the previous ones. We then need to reparametrize the family of quantum states in terms of $\tilde{\boldsymbol{\lambda}}$. Since $\tilde{\partial}_\nu = \sum_\mu B_{\mu\nu}\partial_\mu$ with $B_{\mu\nu} = \partial\lambda_\mu/\partial\tilde{\lambda}_\nu$ we have that

$$\widetilde{L}_\nu = \sum_\mu B_{\mu\nu} L_\mu \tag{21}$$

and the new QFI matrix simply reads

$$\widetilde{\boldsymbol{H}} = B\boldsymbol{H}B^T. \tag{22}$$

We consider here the case where we perform a specific indirect measurement in order to infer the values of the parameters $\boldsymbol{\lambda}$, given some measurement outcomes $\mathcal{X} = \{x_1, x_2, \dots\}$. The whole measurement process can be described by the conditional probability $p(x|\boldsymbol{\lambda})$ of obtaining the value x from the measurement when the parameters have the values $\boldsymbol{\lambda}$. Given this object, we can define the Fisher information (FI) matrix whose elements are obtained as

$$\boldsymbol{F}_{\mu\nu} = \int dx\, p(x|\boldsymbol{\lambda}) \frac{\partial \ln p(x|\boldsymbol{\lambda})}{\partial\lambda_\mu} \frac{\partial \ln p(x|\boldsymbol{\lambda})}{\partial\lambda_\nu}. \tag{23}$$

This matrix defines a bound on the covariance matrix $\boldsymbol{\gamma}$ for the specific measurement we performed. In particular we are interested in the bound for the variance of a single-parameter, at fixed values of the others, which reads

$$\mathrm{Var}(\lambda_\mu) \geq \frac{1}{M}\boldsymbol{F}(\boldsymbol{\lambda})^{-1}_{\mu\mu} \geq \frac{1}{M}\boldsymbol{H}(\boldsymbol{\lambda})^{-1}_{\mu\mu} \tag{24}$$

and which in turn is always lower bounded by quantum CRB given in Eq. (20).
Notice that if we have to reparametrize our family of states in terms of different parameters $\widetilde{\boldsymbol{\lambda}}$, we can use the same formulas shown above for the QFI matrix, obtaining the new FI matrix as $\widetilde{\boldsymbol{F}} = B\boldsymbol{F}B$.

Physical model and evaluation of quantum discord

Here we give explicit expressions of the formulas used in the main text. A two-mode squeezed thermal state (STS) is fully characterized by the two parameters $N_s = \sinh^2 s$ and N_t, representing, respectively, the effective amount of squeezing photons and thermal photons. In our experimental model, these quantities can be obtained as a function of the physical parameters $\{r, \gamma, \eta\}$, that is

$$\begin{aligned} N_s &= \frac{1}{2}\left(-1 + \frac{A(r,\gamma,\eta)}{\sqrt{\eta^2\cosh^4 r\cosh^2(2r\gamma) + B(r,\eta)^2 + 2\eta\cosh^2 r(-2\eta\cosh^4(r\gamma)\sinh^2 r + \cosh(2r\gamma)B(r,\eta))}}\right) \\ N_t &= \frac{1}{2}\left(-1 + \sqrt{(A(r,\gamma,\eta) - \eta\cosh^2(r\gamma)\sinh 2r)(A(r,\gamma,\eta) + \eta\cosh^2(r\gamma)\sinh 2r)}\right) \end{aligned} \tag{25}$$

where

$$A(r,\gamma,\eta) = 1 - \eta + \eta\cosh^2 r\cosh 2r\gamma + \eta\sinh^2 r \tag{26}$$

$$B(r,\eta) = 1 - \eta + \eta\sinh^2 r. \tag{27}$$

Notice that by varying the pump power, we change the parameter r only, while the noise parameters γ and η stay constant to a very good level of approximation (it should depend on the mode matching only). As a result, both the effective squeezing and thermal photons N_s and N_t change accordingly.
The covariance matrix of a two-mode STS can be written as

$$\Sigma_{sts} = \begin{pmatrix} a1_2 & c\,\sigma_z \\ c\,\sigma_z & a1_2 \end{pmatrix} \tag{28}$$

where

$$a = (1+2N_t)(1+2N_s) \tag{29}$$

$$c = 2(1+2N_t)\sqrt{N_s(N_s+1)}, \tag{30}$$

and 1_2 and σ_z are respectively the 2×2 identity matrix and the Pauli matrix for the z direction. Following [22], the quantum discord can be thus evaluated, obtaining

$$\begin{aligned} D(N_s,N_t) = 2N_t\log(N_t) - 2(N_t+1)\log(N_t+1) - (N_s+N_t+2N_sN_t)\log(N_s+N_t+2N_sN_t) + \\ -\frac{N_t(N_t+1)}{1+N_s+N_t+2N_sN_t}\log\left(\frac{N_t(N_t+1)}{1+N_s+N_t+2N_sN_t}\right) + (1+N_s+N_t+2N_sN_t)\log(1+N_s+N_t+2N_sN_t) \\ +\frac{N_s+2N_sN_t+(1+N_t)^2}{1+N_s+N_t+2N_sN_t}\log\left(\frac{N_s+2N_sN_t+(1+N_t)^2}{1+N_s+N_t+2N_sN_t}\right) \end{aligned} \tag{31}$$

expressed as a function of the effective parameters N_s and N_t. By using Eqs. (25), one can easily obtain the discord as a function of the physical parameters r, γ and η.